Geographical effect on small-world network synchronization

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We investigate the geographical effect on the synchronization of small-world oscillator networks. We construct small-world geographical networks by randomly adding links to one- and two-dimensional regular lattices, and we find that the synchronizability is a nonmonotonic function of both the coupling strength and the geographical distance of randomly added shortcuts. Our findings demonstrate that the geographical effect plays an important role in network synchronization, which may shed some light on the study of collective dynamics of complex networks.

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Many systems composed of dynamical units can be properly described by complex networks with nodes representing units and links representing interactions among units [1–6]. A great deal of empirical evidence has revealed that real complex networks share some common features, such as small-world (SW) and scale-free (SF) structural properties [7,8]. Understanding how these properties influence dynamics of real networks is deemed a significant task [3,6]. However, real networks are often embedded in Euclidean geographical space, such as the Internet and neuronal networks. Pure topological representations of complex systems would neglect important physical characteristics, such as the geography-induced interactions among individuals, which, however, may play significant roles in the evolution of the networks and the dynamical processes upon networks. From this perspective, some network models associated with geographical restrictions have been developed for revealing the underlying dynamics of network evolution and for studying dynamical processes with geographical effects [9–11].

Synchronization among dynamical units has been widely observed in complex systems. Typical examples include the flash of fireflies, the sound of hands clapping, the oscillation of chemical reactions, etc. [12–15]. To investigate collective synchronous behaviors, oscillator network models have been commonly used [16]; these are natural representations of real systems consisting of coupled units. In this framework, it has been found that the SW property can remarkably enhance the synchronizability while SF networks tend to inhibit synchronization as compared to regular lattices [17,18]. Moreover, the exploration of which structural property plays the most significant role in synchronization has received much attention [19–21]. Recently, topology-induced weighted coupling has been considered in oscillator networks to better mimic real synchronization behaviors [22–24]. However, the geographical effect on network synchronization has not been considered seriously. Geography-induced coupling differences between units may play a significant role. Both in the flash of fireflies and in hand clapping, the coupling strength between two units is indeed closely correlated with their geographical distance, since the coupling signal in transmissions may be weakened by increase of the distance, particularly for sound and light. In some man-made networks, the coupling strength may be positively correlated with distance. For example, in power grids, longer electrical cables usually carry more electric power. Therefore, there is an obvious need to model oscillator networks by taking the combination of the geographical effect and the coupling strength into account.

In this paper, we investigate the collective synchronization behaviors of oscillators on one- and two-dimensional SW networks with coupling strengths depending on the geographical distance. Our work follows the study of synchronization on weighted networks. Our main result is that the network synchronizability is a nonmonotonic function of both the coupling strength and the geographical distance of the shortcuts, with the strongest synchronizability in the middle of the range. The present study indicates that the geographical effect plays a significant role in the collective synchronization behaviors of SW oscillator networks.

The dynamics of a general weighted network of N coupled identical oscillators has been previously investigated in Refs. [22–24]. Each node of a network represents an oscillator, and a link connecting two nodes represents the coupling between them. Let $\mathbf{x}^i$ be the m-dimensional vector of dynamical variables of the ith node. The following set of equations of motion governs the dynamics of the N coupled oscillators:

$$\dot{\mathbf{x}}^i = \mathbf{f}(\mathbf{x}^i) + c\sum_{j=1}^{N} W_{ij}A_{ij}[\Gamma(\mathbf{x}^j) - \Gamma(\mathbf{x}^i)] = \mathbf{f}(\mathbf{x}^i) + c\sum_{j=1}^{N} G_{ij}\Gamma(\mathbf{x}^j), \qquad (1)$$

where $\dot{\mathbf{x}}^i=\mathbf{f}(\mathbf{x}^i)$ describes the dynamics of an individual oscillator, $\Gamma(\mathbf{x})$: $\mathbb{R}^m \to \mathbb{R}^m$ is a vectorial output function, c is the overall coupling strength, W_{ij} is the link weight, $A=(A_{ij})$ is the adjacency matrix, and $G=(G_{ij})$ is the coupling matrix: $G_{ij}=\delta_{ij}S_i - W_{ij}A_{ij}$ where $S_i=\sum_{j=1}^{N} W_{ij}A_{ij}$. The adjacency matrix A is binary and both A and G are symmetric.

If the network (1) is diffusively and irreducibly coupled, then all the eigenvalues of matrix G are nonpositive real values because G is negative semidefinite, and its biggest eigenvalue γ_0 is always zero because the rows of G have

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zero sum. Thus, the eigenvalues can be ranked as $0=\gamma_0 > \gamma_1 \geq \cdots \geq \gamma_{N-1}$, and the synchronization manifold is an invariant manifold, with a fully synchronized state: $\mathbf{x}^1=\mathbf{x}^2=\cdots=\mathbf{x}^N=\mathbf{s}$ satisfies $\dot{\mathbf{s}}(t)=\mathbf{f}(\mathbf{s}(t))$.

Let ξ_i be the variation on the ith node, and linearize (1) about $\mathbf{s}(t)$ [25] to get the variational equation

$$\dot{\xi}_k = [D\mathbf{f}(s) + c\gamma_k D\Gamma(s)]\xi_k, \tag{2}$$

where $D\mathbf{f}(s)$ and $D\Gamma(s)$ denote the Jacobian matrices of $\mathbf{f}(s)$ and $\Gamma(s)$, respectively, and $k=1,2,\ldots,N-1$. Let $\epsilon=c\gamma_k$ and $\Delta=[\xi_1,\xi_2,\ldots,\xi_N]$, and rewrite (2) as

$$\dot{\Delta} = [D\mathbf{f}(s) + \epsilon D\Gamma(s)]\Delta. \tag{3}$$

In the symmetric coupling setting (namely, $G_{ij}=G_{ji}$), since $D\mathbf{f}(s)$ and $D\Gamma(s)$ are the same for each block, the largest Lyapunov exponent $\lambda_{\max}$ of (3) depends only on ϵ. The sign of the master stability function determines the synchronized state: the synchronized state is stable if $\lambda_{\max}(\epsilon)<0$ for all blocks [25].

For many dynamical systems, if the master stability function is negative in a single finite interval (ϵ_1,ϵ_2) then the largest Lyapunov exponent is negative. Therefore, the network is synchronizable for some c when the eigenratio $R=\gamma_{N-1}/\gamma_1$ satisfies

$$R \equiv \gamma_{N-1}/\gamma_1 < \epsilon_2/\epsilon_1. \tag{4}$$

In (4), the eigenratio R depends only on the network structure and the range from ϵ_1 to ϵ_2 depends on the node dynamics. From this, it follows that the smaller the eigenratio R, the more synchronizable the network, and vice versa [25].

In order to introduce the notion of geographical distance into SW networks, we adopt a modified SW network model, generated from one- and two-dimensional regular lattices, through randomly adding shortcuts to the original lattices. Here, for simplicity, periodic boundary conditions are assumed for both regular lattices. For a d-dimensional lattice, $d=1,2$ here, each node i has d Euclidean geographical coordinates $(a_1^i,a_2^i,\ldots,a_d^i)$. The geographical distance L_{ij} between nodes i and j is defined as [26]

$$L_{ij} = \sum_{k=1}^{d} |a_k^i - a_k^j|. \tag{5}$$

This definition is also called the Manhattan distance [26]. A more detailed description of the geographical distance can be seen in Fig. 1. Then one can calculate the geographical distance in a given d-dimensional lattice. To add shortcuts with fixed distance L, node i was chosen randomly and then another node j was chosen in an equally random way from the nodes whose geographical distance from i equaled L. If there was no link between the two nodes, then a shortcut was established; otherwise another node was selected as j. This process was repeated until m new links were created. Thus a d-dimensional SW network was obtained and all added shortcuts had identical geographical distance L.

The distance-induced coupling strength between two connecting nodes i and j is defined as

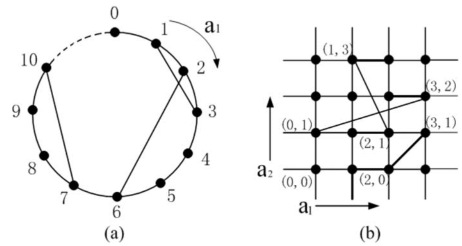

FIG. 1. (a) One-dimensional ring graph and (b) two-dimensional lattices with some shortcuts. a_1 and a_2 are the coordinate directions. In (a), nodes are marked by their coordinates. The geographical distance L between each pair of nodes can be calculated by Eq. (5). The geographical distance of a link is the geographical distance between the two nodes at the ends of this link. For instance, the link geographical distance $L_{1,3}$ between nodes 1 and 3 is $|1-3|=2$. Analogously, $L_{2,6}=|2-6|=4$, and $L_{7,10}=|7-10|=3$. In (b), some nodes are marked by their coordinates (a_1,a_2). We can calculate the geographical distance between each pair of nodes and the geographical distance of each link according to Eq. (5). For instance, the link geographical distance $L_{(2,0),(3,1)}$ between nodes (2,0) and (3,1) is $|2-3|+|0-1|=2$. Analogously, $L_{(1,3),(2,1)}=|1-2|+|3-1|=3$, $L_{(0,1),(3,2)}=|0-3|+|1-2|=4$, and $L_{(0,0),(2,0)}=|0-2|+|0-0|=2$. Note that the geographical distance is independent of the position of the origin (0,0). The unit geographical distance between two geographically neighboring nodes is 1. The range for the spatial coordinate is determined by both the dimension and the network size. The periodic boundary condition is used here.

$$W_{ij} = W_{ji} = L_{ij}^{\alpha}, \tag{6}$$

where L_{ij} is defined by (5) and α is a tunable parameter which governs the intensity of the coupling strength. Here, the coupling strength between two geographically neighboring nodes is always 1, regardless of the value of α. In the following, we focus on the influences of the parameter α and the geographical distance L of the shortcuts on the collective synchronization dynamics.

For both one- and two-dimensional SW networks, we set the network size to 2025 in all simulations. Each data point is obtained by averaging over 100 independent realizations. Figure 2 shows the eigenratio R as a function of the tunable parameter α for different geographical distance L. Here, for easy comparison, the number of added links m is fixed to 400 for all model parameters. These links are enough to introduce small-world properties in the network with size 2025. One can see that the strongest synchronizability occurs at the point of $\alpha=0$, independent of the value of L. The synchronization behavior in the range of $\alpha<0$ can be easily understood: according to (6), the lower the value of α, the weaker the coupling strength among oscillators, which leads to worse synchronizability. When α approaches negative infinity, the coupling strengths of those shortcuts L^{α} are approximately zero and the SW network is reduced to the original unweighted regular lattice. It is well known that the synchronizability is considerably inhibited in regular lattices as compared to SW networks, which is consistent with our simulation results. In the range of $\alpha>0$, the change of the synchronizability as α increases may be counterintuitive, i.e.,

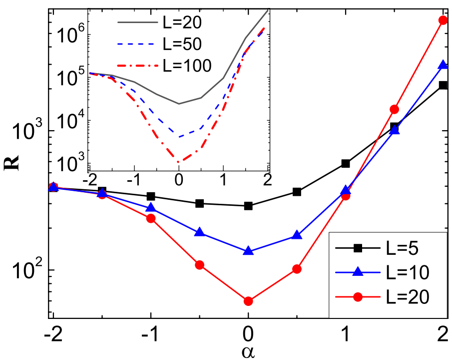

FIG. 2. (Color online) Eigenratio R of two-dimensional SW networks as a function of the tunable parameter α for different geographical distance L of the shortcuts. Other parameters: network size N=2025 (width=45), number of added links m=400. The inset shows the similar results of one-dimensional SW networks (network size N=2025). All the results are averaged over 100 different network realizations.

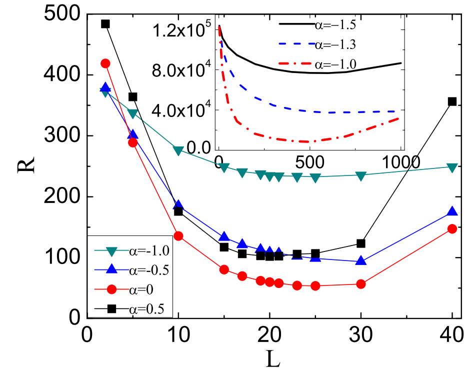

FIG. 3. (Color online) Eigenratio R of two-dimensional SW networks as a function of the geographical distance L of the shortcuts for different values of parameter α. Other parameters: network size $N=45\times45=2025$, number of added links m=400. The inset shows the results of one-dimensional SW networks (network size N=2025). All the results are averaged over 100 different network realizations.

a stronger coupling strength results in a worse synchronizability. This fact can be partially explained from the viewpoint of intensity heterogeneity [22,24]; it was found that the heterogeneous intensity of nodes restrains the synchronizability of weighted SF networks. In other words, a weighted network with more homogeneous intensity can synchronize more easily. Similarly, for weighted SW networks, the node intensity seems to play the same role in synchronization. For positive values of α, as α increases, the coupling strengths of the shortcuts become stronger, so that the intensity differences between the nodes at the ends of the shortcuts and the nodes without shortcut connections become bigger and bigger. Hence, the network becomes more difficult to synchronize with increase of α, reflected by the enhancement of the eigenratio R. The inset of Fig. 2 shows the results for one-dimensional SW networks, which are analogous to the two-dimensional case.

Next, we study the eigenratio R as a function of the geographical distance L of the shortcuts for different fixed values of the coupling parameter α. As shown in Fig. 3, as L increases from 2 to about 25, the eigenratio decreases rapidly and reaches a minimum value, indicating that the strongest synchronizability is achieved. Thereafter, the value of R becomes larger when L of the shortcuts is continuously increased. A similar varying trend of R was found in one-dimensional SW networks, as displayed in the inset of Fig. 3. This finding may not be easily understood. In geographically embedded networks, the topological and geographical distances have a rough relationship. Generally, if the geographical restriction is strong, i.e., connections are established more locally (L is small), the average topological distance and the diameter are large, due to the absence of long-range connections; otherwise, more long-geographical-distance connections usually reduce the diameter of the network [11], which would benefit the synchronization of the network [17]. However, this expectation is inconsistent with the result in Fig. 3. To explain the nonmonotonic phenomenon, we further investigated the average topological distance $\langle l\rangle$ as a function of the geographical distance L of the shortcuts in two-dimensional SW networks. As shown in Fig. 4, the curves of $\langle l\rangle$ against L resemble that in Fig. 3. Hence, it is the increment of $\langle l\rangle$ that leads to the nonmonotonic behavior of synchronizability, whereas the degrees of nodes are highly homogeneous in SW networks in particular.

In summary, we have studied the geographical effect on collective synchronization behaviors by combining the notions of geographical distance and the coupling strength on geographical small-world networks. We have focused on the synchronizability affected by both the coupling parameter α and the distance L of the added shortcuts. We found that the synchronizability is a nonmonotonic function of both α and L on both one- and two-dimensional small-world networks. These results indicate that the geographical effect plays an important role in collective network synchronization behaviors. We have also provided some qualitative explanations

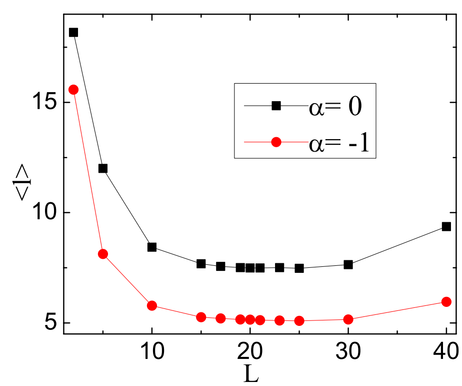

FIG. 4. (Color online) Average topological distance $\langle l\rangle$ of two-dimensional SW networks with periodic boundary conditions as a function of the geographical distance L for different values of parameter α. $\langle l\rangle$ is defined as the average shortest hops between each pair of nodes [7]. Other parameters: network size N=2025 (width =45), number of added connections m=400. Here, for $\alpha=-1$, $\langle l\rangle$is the weighted topological distance. The weights of all links are their geographical distances. All the results are averaged over 100 independent runs.

for the network synchronizability in terms of the heterogeneity of the node intensity and the average topological distance. Since the geographical distance is an important parameter of many real networks, our findings shed some light on the collective dynamics of real coupled systems.

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002).
[3] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[4] D. J. Watts, Annu. Rev. Sociol. **30**, 243 (2004).
[5] R. Pastor-Satorras and A. Vespignani, *Evolution and Structure of the Internet* (Cambridge University Press, Cambridge, U.K., 2004).
[6] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[7] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[8] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[9] S.-H. Yook, H. Jeong, and A.-L. Barabási, Proc. Natl. Acad. Sci. U.S.A. **99**, 13382 (2002); A. F. Rozenfeld, R. Cohen, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **89**, 218701 (2002).
[10] L. Huang, L. Yang, and K. Yang, Phys. Rev. E **73**, 036102 (2006); X.-J. Xu, W.-X. Wang, T. Zhou, and G. Chen, Int. J. Mod. Phys. C **17**, 1815 (2006); X.-J. Xu, Z.-X. Wu, and G. Chen, Physica A **377**, 125 (2007).
[11] Y. Hayashi, IPSJ Digital Courier **2**, 155 (2006).
[12] S. H. Strogatz and I. Stewart, Sci. Am. **269**, 102 (1993).
[13] S. Bottani, Phys. Rev. E **54**, 2334 (1996).
[14] L. Glass and M. C. Mackey, *From Clocks to Chaos: The Rhythms of Life* (Princeton University Press, Princeton, NJ, 1988).
[15] Z. Néda, E. Ravasz, Y. Brechet, T. Vicsek, and A.-L. Barabási, Nature (London) **403**, 849 (2000).
[16] D. J. Watts, *Small Worlds* (Princeton University Press, Princeton, NJ, 1999); L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A. Sigüenza, Phys. Rev. Lett. **84**, 2758 (2000); X. F. Wang, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 885 (2002); M. Timme, F. Wolf, and T. Geisel, Phys. Rev. Lett. **89**, 258701 (2002); J. Ito and K. Kaneko, Phys. Rev. E **67**, 046226 (2003); F. M. Atay, J. Jost, and A. Wende, Phys. Rev. Lett. **92**, 144101 (2004); Y. Moreno and A. F. Pacheco, Europhys. Lett. **68**, 603 (2004); P. G. Lind, J. A. C. Gallas, and H. J. Herrmann, Phys. Rev. E **70**, 056207 (2004); L. Huang, K. Park, Y.-C. Lai, L. Yang, and K. Yang, Phys. Rev. Lett. **97**, 164101 (2006).
[17] X. Guardiola, A. Diaz-Guilera, M. Llas, and C. J. Perez, Phys. Rev. E **62**, 5565 (2000).
[18] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
[19] H. Hong, B. J. Kim, M. Y. Choi, and H. Park, Phys. Rev. E **69**, 067105 (2004).
[20] C.-Y. Yin, W.-X. Wang, G. Chen, and B.-H. Wang, Phys. Rev. E **74**, 047102 (2006).
[21] M. Zhao, T. Zhou, B.-H. Wang, and W.-X. Wang, Phys. Rev. E **72**, 057102 (2005).
[22] A. E. Motter, C. Zhou, and J. Kurths, Phys. Rev. E **71**, 016116 (2005).
[23] M. Chavez, D.-U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett. **94**, 218701 (2005).
[24] C. Zhou, A. E. Motter, and J. Kurths, Phys. Rev. Lett. **96**, 034101 (2006).
[25] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998); M. Barahona and L. M. Pecora, *ibid.* **89**, 054101 (2002).
[26] B. K. Singh and N. Gupte, Phys. Rev. E **71**, 055103(R) (2005).