Mutual attraction model for both assortative and disassortative weighted networks

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For most complex networks, the connection between a pair of nodes is the result of their mutual affinity and attachment. In this letter, we will propose a mutual attraction model to characterize weighted evolving networks. By introducing the initial attractiveness A and the general mechanism of mutual attraction (controlled by parameter m), our model can naturally reproduce scale-free distributions of degree, weight, and strength, as found in many real systems. Also, simulation results are consistent with theoretical predictions. Interestingly, we obtain nontrivial clustering coefficient C and tunable degree assortativity r, depending on the values of m and A. Our model appears as a more general one that unifies the characterization of both assortative and disassortative weighted networks.

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The past few years have witnessed a great deal of interest from the physics community to understand and characterize the underlying mechanisms that govern the evolution of complex networks [1–3]. Prototypical examples cover as diverse as the Internet [4], the World Wide Web [5], the scientific collaboration networks (SCN) [6,7], and worldwide airport networks (WAN) [8,9]. As a landmark, Barabási and Albert (BA) proposed their seminal model that introduced the linear preferential linking to mimic the topological evolution of complex networks [10]. However, networks are far from a Boolean structure. The purely topological characterization will miss important attributes often encountered in real systems. For instance, the amount of traffic characterizing the connections of communication systems or large transport infrastructure is fundamental for a full description of these networks [11]. This thus calls for the use of weighted network representation, which is often denoted by a weighted adjacency matrix with element w_{ij} representing the weight on the edge connecting vertices i and j. In the case of undirected graphs, weights are symmetric $w_{ij}=w_{ji}$, as we will focus on. A natural generalization of connectivity in the case of weighted networks is the vertex strength defined as $s_i=\Sigma_{j\in\Gamma(i)}w_{ij}$, where the sum runs over the set $\Gamma(i)$ (neighbors of node i). This quantity is a natural measure of the importance or centrality of a vertex in the network. Most recently, the access to more complete empirical data and higher computation capability allowed scientists to study the variation of the connection weights of many real graphs. As confirmed by measurements, complex networks not only exhibit a scale-free degree distribution $P(k)\sim k^{-\gamma}$ with $2\leqslant\gamma\leqslant 3$ [8,9], but also the power-law weight distribution $P(w)\sim w^{-\theta}$ [12] and the strength distribution $P(s)\sim s^{-\alpha}$ [9]. Highly correlated with the degree, the strength usually displays scale-free property $s\sim k^{\beta}$ with $\beta\geqslant 1$ [9,13,14]. Motivated by all those findings, Barrat *et al.* presented a model (BBV for short) to study the growth of weighted networks [15]. Controlled by a single parameter δ, the BBV model can produce scale-free properties of degree, weight, and strength. But its disassortative property [15,16] (i.e., the hubs are primarily connected to less connected nodes), as observed in real technological and biological networks, can hardly give satisfying interpretations to social networks like the SCN where the hubs are very likely to be linked together (i.e., assortative mixing). Previous models, to the best of our knowledge, can generate either assortative networks [17–19] or disassortative ones [15–17,20], but rarely both. Thus, some questions arise here: why are social networks all assortative, while all biological and technological networks opposite? Is there a generic explanation for the observed incompatible patterns, or does it represent a feature that needs to be addressed in each network individually? Our work may shed some light to these questions.

Former network models often impress on people such an evolution picture: preexisting nodes are passively attached by newly added ones according to the preferential linking mechanism. This scenario, however, lacks the other side of the fact that old nodes will choose the young at the same time. In addition, this evolution picture also ignores the universal mutual attraction between existing components, which leads to the creation and reinforcement of connections. This idea has been partly reflected in the studies of Dorogovtsev and Mendes (DM) [22] who proposed a class of undirected and unweighted models where new edges are added between old sites and existing edges can be removed. In this paper, we will present a model to study the weighted network evolution under the general mechanism of mutual attraction between nodes. In contrast with previous models where weights are assigned statically [23,24] or rearranged locally [15,16], our model allows weights to be widely updated. It can mimic the reinforcement and creation of internal links as well as the evolution of many infrastructure networks. Specifically, the model can generate a diversity of scale-free quantities, nontrivial clustering property, and tunable assortativity coefficient, in good accord with the features of various real networks.

The model starts from $N_0=m$ isolated nodes, each with initial attractiveness A. At each time step, a new isolated node n is introduced into the system. Then, every existing

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node i preferentially selects m other nodes with the probability

$$\Pi_{i\to j}=\frac{s_j+A}{\sum_{k(\neq i)}(s_k+A)}. \tag{1}$$

Such selection is totally free and does not guarantee the creation of new links or an increase of edge weights between node pairs. Unless two nodes mutually select each other (in other words, unless they attract), there will be no change to the pair of nodes or their connection. If they do, then the weight of their link w is supposed to increase by 1. As a remark, w can be regarded as 0 if the nodes were not connected before, and the mechanism is globally implemented for all the nodes. After updating all the edge weights and node strengths, the growth process is iterated by introducing a new node, until the desired size of the network is reached. Obviously, to guarantee the isolated nodes can be chosen by others, the model requires $A>0$ which governs the probability for "young" nodes to get new links and weights [25].

From our model mechanism, it is easy to seek appropriate interpretations to real networks. Take the SCN, for example: the collaboration of scientists requires their common interest and mutual acknowledgments. Unilateral effort does not make effective activity. In terms of model language, though the low-degree nodes like to connect to the large-degree ones, the latter do not necessarily wish to be linked by the former. On the other hand, two scientists both with strong scientific potential and a long collaborating history are more likely to write papers together. The above description of our model may also satisfactorily explain the WAN, where the edge weight denotes the relative magnitude of the traffic along a flight line. During the evolution of WAN, the airlines are more likely to open between metropolises that hold high status in both economy and politics (with large strengths). With development of economy and population expansion, the air traffic between connected metropolises will increase much faster than that between smaller towns. Due to their importance, there is an obvious need for smaller towns to bridge those metropolises. But the limit of energy and resources leaves the fact that each site can only afford a finite number of connections. Therefore, they have to choose in front of the vertex pool. For technological networks with traffic taking place on them, the mutual attraction may be caused by both the limit of resources and the internal demand of traffic increment for maintaining the normal functioning of the networks [20,21].

One may find that the mathematical structure of the present model is very close to our previous models [20,21], though different in model description. It is still worth noticing that our previous models are generally defined by two model parameters: one describes the increasing rate of internal traffic like W in Ref. [20] or m in Ref. [21], while the other captures the external growth of the network. As we stated in our original paper [20], the generated networks can only describe technological networks such as the Internet due to their disassortative mixing property. Differently, the current model is defined based on the assumption of uniform initial attractiveness A and the maximum of potential connection m, which give rise to the tunable assortative mixing patterns of weighted networks. Thus, it may be used to interpret both assortative and disassortative patterns emerging in complex networks. Anyway, considering their similarity in mathematics, one may regard the current model as a meaningful and generalized extension of our previous work.

The model time is measured with respect to the number of nodes added to the graph, i.e., $t=N-N_0$, and the natural time scale of the model dynamics is the network size N. Using the continuous approximation, we can treat k,w,s, and the time t as continuous variables [10,15]. Considering the rule that w_{ij} is updated only if node i and j select each other, the time evolution of weight can be computed analytically as follows:

$$\begin{aligned}\frac{dw_{ij}}{dt}&=m\frac{s_j+A}{\sum_{k(\neq i)}(s_k+A)}\times m\frac{s_i+A}{\sum_{k(\neq j)}(s_k+A)}\\&\approx\frac{m^2(s_i+A)(s_j+A)}{\sum_k(s_k+A)\sum_k(s_k+A)}.\end{aligned} \tag{2}$$

Hence, the strength $s_i(t)$ is updated by the rate

$$\frac{ds_i}{dt}=\sum_j\frac{dw_{ij}}{dt}\approx\frac{m^2(s_i+A)}{\sum_k(s_k+A)}=\frac{m^2(s_i+A)}{(m^2+A)t}. \tag{3}$$

The last expression is recovered by noticing that

$$\sum_i s_i=\int_0^t\frac{d\sum_i s_i}{dt}dt=\int_0^t\frac{m^2\sum_i(s_i+A)}{\sum_k(s_k+A)}dt=m^2t.$$

From Eq. (3), one can obtain the scaling of $s_i(t)$ versus t as $s_i(t)\sim t^{\lambda}$, which also implies the scale-free distribution of strength $P(s)\sim s^{-\alpha}$ with the exponent [15] $\alpha=1+1/\lambda=1+(m^2+A)/m^2=2+A/m^2$. One can also obtain the evolution behaviors of weight and degree, and hence their power-law distributions: $P(w)\sim w^{-\theta}$ with $\theta=2+2A/(m^2-A)$ and $P(k)\sim k^{-\gamma}$ with $\gamma\to 2+A/m^2=\alpha$ as $t\to\infty$ Ref. [26].

We performed numerical simulations of networks generated by choosing different values of A and m. The results well recover the above theoretical predictions. Figures 1(a)–1(d), fixed $A=1$ and tuned by m, report the probability distributions of strength, weight and degree, as well as the strength-degree correlation. Specifically, Fig. 1(a) gives the probability distribution $P(s)\sim s^{-\alpha}$, which is in good agreement with the theoretical expression. Probability weight distribution also recovers the power-law behavior $P(w)\sim w^{-\theta}$ [Fig. 1(b)] with θ as predicted analytically. Figure 1(c) shows the scale-free degree distribution $P(k)\sim k^{-\gamma}$ and Fig. 1(d) reports the average strength of vertices with degree k_i, which displays a nontrivial power-law behavior $s\sim k^{\beta}$. For $m=10$ and $N=10000$, β is near 1.5, as the empirical finding in worldwide airport network [9]. The inset of Fig. 1(d) indicates that the exponent β decreases very slowly with the network size, which is noticeably different from the linear correlation ($\beta=1$) as obtained in most previous models.

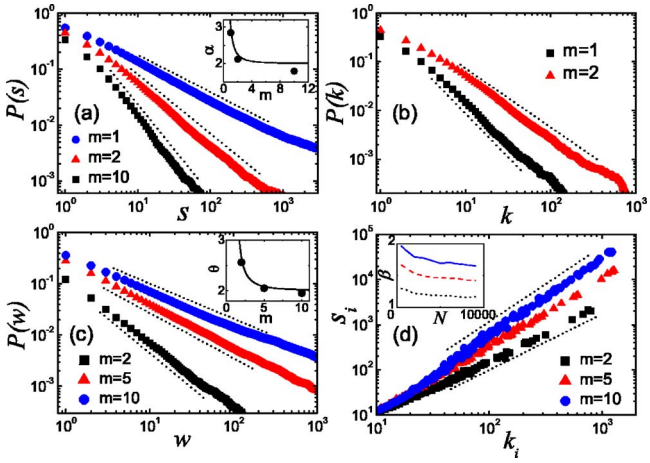

FIG. 1. (Color online) Numerical simulations by choosing $A=1$. Data are averaged over 10 independent runs of network size $N=8000$: (a) Cumulative probability strength distribution $P(s)$ with various m. Data are consistent with a power-law behavior $P(s)\sim s^{-\alpha}$. The inset reports the values of α obtained by data fitting (full circles) in comparison with the theoretical prediction $\alpha=2+A/m^2$ (line). (b) Cumulative probability degree distribution $P(k)$ with $m=1$ and $m=2$. Data fitting confirms its scale-free property. (c) Cumulative probability distribution of weight with different m, in agreement with the power-law tail $P(w)\sim w^{-\theta}$. As shown in its inset, the data fitting also gives values of θ (full circles) as predicted by analytical calculation (line). (d) The average strength s_i of nodes with connectivity k_i for different m. In the log-log scale, we observe the nontrivial strength-degree correlation $s\sim k^{\beta}$, with the exponent β versus network size N (see the inset).

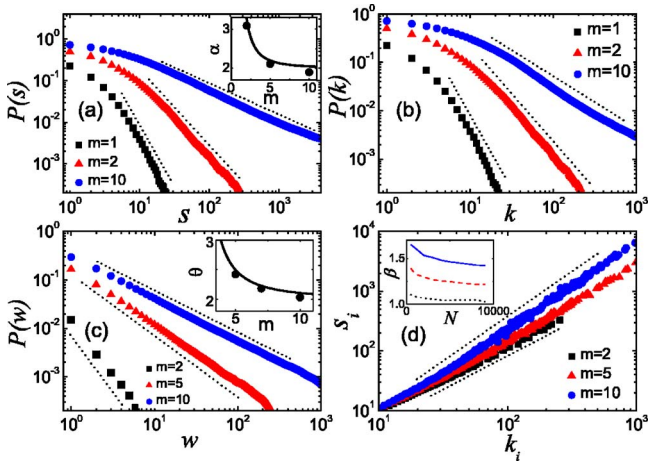

FIG. 2. (Color online) Numerical simulations by choosing $A=5$. Data are averaged over 10 independent runs of network size $N=8000$: (a) $P(s)\sim s^{-\alpha}$ with different m. The inset reports α obtained by data fitting (full circles) in comparison with the theoretical prediction (line). (b) $P(k)\sim k^{-\gamma}$ by choosing different m. Data fitting confirms its scale-free property. (c) $P(w)\sim w^{-\theta}$. The data fitting in the inset also gives values of θ (full circles) as predicted analytically (line). (d) The strength s_i vs connectivity k_i. In the log-log scale, we observe the nontrivial strength-degree correlation $s\sim k^{\beta}$, with β vs network size N shown in its inset.

Again, Figs. 2(a)–2(d) show the simulation results by fixing a moderate value of $A=5$ and varying m. In comparison with Figs. 1(a) and 1(b), the distributions of strength and degree for $A=5$ both behave as exponential corrections in the zone of low degree. This phenomenon occurs at large A and the exponential parts are interestingly very similar with the empirical findings in some social networks like SCN [27]. The larger the initial attractiveness A (vs m), the larger the effect of exponential correction at the head. In the zone of large degree, however, we can still observe the power-law tail which again recovers the theoretical exponent expressions. It is worth remarking that the model at large A can generate the assortative property too, which is a special feature of social networks. Therefore, the introduction of A is essential for our model to mimic social networks.

To better understand the architecture structure and degree correlations of the model networks, we also studied the unweighted clustering coefficient C (which describes the statistic density of connected triples [27,28]) and degree assortativity r [29] depending on the model parameters A and m. As presented in Fig. 3(a), C for fixed m monotonously decreases with A, and C for fixed A monotonously increases with m. Generally, it can be tuned in the range $]0,1[$. The clustering property of our model is tunable in a broad range by varying both m and A, which makes it more powerful in modeling real networks. As shown in Fig. 3(b), degree assortativity r for fixed m, unlike the clustering case, increases with increasing A, while r for given A decreases with m. For small A and large m, the model generates disassortative networks which can best mimic technological networks like the Internet [4] and WAN or even biological networks. While at large A and small m, assortive networks emerge and can be used to model social graphs as the SCN. Actually, enhancing the initial attractiveness A will considerably increase the chances for "young" nodes to be linked and strengthened. Since low-degree nodes take the majority in the system, larger A will lead to the stronger affinity between "young" vertices, and thus they can link together more easily. This explains the origin of assortative mixing in our model and may also shed some light on the old open question: why are social networks different from other networks in degree assortativity? The components of social networks are human beings who share a complex nature. Take the SCN, for example; the attractiveness of a scientist could not be represented simply by his (or her) total publications (as some previous models indicated [15]), i.e., the strength of node in SCN. Actually, there are

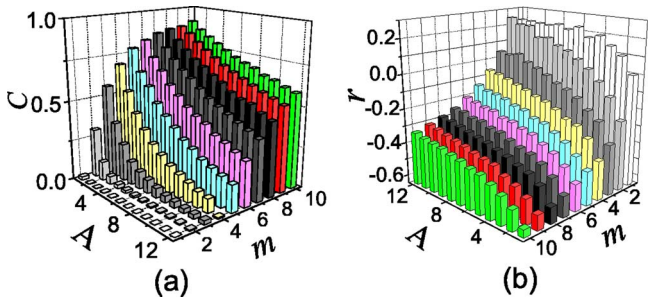

FIG. 3. (Color online) (a) Clustering coefficient C depending on both m and A with $N=8000$. (b) Degree assortativity r depending on both m and A with $N=8000$.

many other important qualities that will contribute to the attractiveness of a scientist, for instance, his temper, scientific environment, and social ability, etc. All of such elements are not reflected in the item of node strength, but might be integrated in our model as the introduction of initial attractiveness. Indeed, due to the indescribable complexity of human beings in the social networks, one can hardly give a general explanation of the emergence of its assortative mixing in contrast with the technological networks. Perhaps the different initial attraction contributes to their fundamental differences. On the other end, as m controls the interaction frequency among the network internal components, increasing m will make the hubs become busier and busier, as they have to be linked by more and more "young" sites. It may explain why the disassortativity of the model is increasingly sensitive to m. In addition, the components of technological networks are usually physical devices, interacting with each other under standard technical rules. All too often, the node strength (i.e., the total traffic a site handles) can already reflect their importance or attractiveness. So in this case, it appears hard to appreciate the significance of initial attractiveness. Combining these two parameters together, our current model integrates two competitive ingredients that may be responsible for the mixing difference in complex networks.

In sum, the general dynamics of node interaction proposed here produces a wide variety of scale-free behaviors, nontrivial clustering, and degree assortativity. Our current model may mimic both the assortative and disassortative networks under a unified evolution dynamics. Its obvious simplicity and reproduced real-world variety will allow more specific mechanisms to be integrated into future modeling work, and it may also be meaningful for understanding physics processes (such as traffic congestion and synchronization) on real networks which are both weighted and correlated. Most importantly, the model may indicate the possible and worthwhile efforts in exploring the simplified and unified mechanisms behind various complex networks.

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