Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Cascading failure spreading on weighted heterogeneous networks

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Received 12 March 2008
Accepted 5 May 2008
Published 29 May 2008

Online at stacks.iop.org/JSTAT/2008/P05013
doi:10.1088/1742-5468/2008/05/P05013

Abstract. We study the onset and spreading of cascading failure on weighted heterogeneous networks by adopting a local weighted flow redistribution rule, where the weight and tolerance of a node is correlated with its link degree k as k^θ and Ck^θ, respectively. The weight parameter θ and tolerance parameter C are positive: $\theta > 0$ and $C > 1.0$. Assume that a failed node leads only to a redistribution of flow passing through it to its nearest-neighboring nodes. We give our theoretical estimations of the onset of the cascading failure for different values of θ. We also explore the statistical characteristics of the avalanche size on the networks by varying θ and obtain versatile dynamical scenarios of the cascading processes, which exhibit either 'subcritical', or 'critical', or 'supercritical' behaviors, depending on the value of the weight parameter. Moreover, we investigate the effect of degree correlation on the cascading failure spreading on weighted scale-free networks. It is found that single failure on assortative scale-free networks may be more likely to induce a series of subsequent failure events up to a large scale, and the cascading dynamics may easily self-organize to a 'criticality' state on such assortative scale-free networks.

Keywords: network dynamics, random graphs, networks

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Contents

1. Introduction

Network science focusing on the relationship between structure, function and dynamics has attracted much attention in recent years [1]–[4]. One of the hot topics in network science is the robustness characteristic of a complex network against attacks and random failures [1]–[4], as it is known that our daily life is closely related to various types of networks, such as power grids, information communication networks, transportation networks, etc. Evidence has demonstrated that in such networks, even though intentional attacks or random failures emerge very locally, the entire network can be greatly affected, often resulting in large scale collapse and unfunctioning of part or the whole of the system. An often cited realistic example is the 1996 blackout of the power transportation network in the USA [5].

Up to now, it has been shown again and again that the topological features of underlying interaction networks have great impacts on the final outcomes of the dynamics taking place on them [1]–[4]. For example, the scale-free topology of a network resulting in a vanishing threshold of an epidemic spreading on it with the increase of the network size, also gives rise to a robust behavior against random failures and fragility for aimed attacks [1, 3]. In fact, understanding how the structure affects the dynamics is regarded as one of the major objectives in the study of complex networks [2]. For this reason, most previous cascading models confined on complex networks, such as the sandpile model [6], the global load-based cascading model [7]–[13] and the fiber bundle model [14]–[16], have also been subjected to this issue. However, the network weights have not been taken into consideration in these models, regardless of the fact that real networks display a large heterogeneity in the weights which have a strong correlation with the network topology [17]–[19]. In fact, the existing common weighted features play a significant role in a variety of dynamical processes, including epidemic spreading, information packet routing, synchronization, etc [20]–[23]. In particular, Wang *et al* [20] introduced a local-information-based routing strategy on scale-free networks, wherein each node is selected as a router by its neighbors with a probability proportional to its weight k^{α}. They showed that, in the case of $\alpha = -1.0$, the whole network achieves its optimal performance. Another influential work concerning alternative routing strategy on networks is studied by Yan *et al* [22], in which each node contributes a weight of k^{β} to any path going through it.

doi:10.1088/1742-5468/2008/05/P05013

It was found at $\beta = 1.0$ the network capability in processing traffic is improved more than 10 times as compared to shortest path routing. More recently, Korniss [23] proposed that complex networks are easy to be synchronized if the coupling strength (denoted by the weight of the edges) between any two connected nodes with degrees k_i and k_j is weighted as $(k_i k_j)^{-1.0}$. Hence, there is a need for a modeling approach that can capture the coupling of cascading and weighted characteristics.

In a recent work of Wang and Chen [24], a cascading model with a local weighted flow redistribution rule (LWFRR) is proposed and studied on weighted scale-free and small-world networks. In their model, the cascading process is triggered by a small initial perturbation (a randomly selected edge with weight $(k_i k_j)^{\theta}$ is attacked) and spreads to other constituents sequentially. They found that the weighted complex network reaches the strongest robustness level at a universal value of the weight parameter $\theta = 1$. In the present work, we follow the research of [24] by considering a cascading model on weighted complex heterogeneous networks, wherein the cascading process is triggered by small initial attacks or failures on network nodes. It is found that, in the node-based cascading model, the condition for universal robustness of the network is unchanged. In addition, by exploring the statistical characteristics of the avalanche size, we obtain versatile scenarios of the cascading dynamical processes, which exhibit either 'subcritical', 'critical' or 'supercritical' behaviors, depending on the value of the weight parameter. Moreover, the degree correlation among nodes is found to have a considerable impact on the cascading failure spreading on weighted scale-free networks.

2. The model

It has been proposed that many realistic networks share two common properties: small average path lengths among any two nodes and a power-law degree distribution [1]–[4]. For simplicity, we use the well-known Barabási–Albert (BA) scale-free network model [1] as the physical infrastructure on top of which a cascading process takes place. The BA model containing two generic mechanisms of many real complex systems—growth and preferential attachment [1]—can be constructed as follows. Starting from m_0 nodes, one node with m links is attached at each time step in such a way that the probability $\prod_i$ of being connected to the existing node i is proportional to the degree k_i of that node, i.e. $\prod_i = k_i / \sum_j k_j$, where j runs over all existing nodes. In the present work, the total network size is fixed as $N = 5000$ and the parameters are set to be $m_0 = m = 2$ (hence the average connectivity of the network is $\langle k \rangle = 2m = 4$ [1]).

With the heterogeneous networks at hand, let us define the cascading model under the LWFRR based on node failure. We assume that the weight of a node i is given by $w_i = k_i^{\theta}$, where θ is a tunable parameter in our study, which controls the strength of the node weight. This assumption is reasonable since many previous studies concerning both model networks and real networks have shown that the load of (or traffic handled by) a node scales with its degree as $L(k) = bk^{\eta}$, where η relies on topological elements [9, 12, 17], [25]–[27]. Following [24], we assume that the potential cascading failure is triggered by a small initial attack or perturbation, e.g. unfunctioning of a single node i. The flow supposed to be going through the broken node i, denoted by F_i, will be redistributed to its nearest-neighboring nodes (see figure 1 for illustration). The additional flow ΔF_j

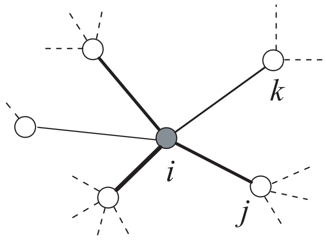

Figure 1. Schematic illustration of the LWFRR triggered by a node failure. The focal node i is broken and the flow along it is redistributed to its neighboring nodes $j, k, \ldots$. Among these neighbors, the one with higher flow capacity (i.e. more degrees) will undertake more shared flow (denoted by the width of the links) from the failed node.

received by the neighboring node j is proportional to its weight:

$$\Delta F_j = F_i \times \frac{w_j}{\sum_{l \in \Omega_i} w_l}, \tag{1}$$

where Ω_i is the set of neighboring nodes of i. Following previous models addressing cascading problems [8, 10, 13, 16, 24], each node i in the network has a weight threshold or capacity $\mathcal{F}_i$, which is the maximum flow that the node can handle. Conventionally, it is assumed that the threshold of a node is proportional to its weight $\mathcal{F}_i = Cw_i = Ck_i^{\theta}$, where the constant $C > 1$ is a threshold parameter characterizing the tolerance of the network [7, 8, 10, 11, 13, 16, 24]. Considering a neighbor j of the node i, if

$$F_j + \Delta F_j > \mathcal{F}_j = Ck_j^{\theta}, \tag{2}$$

then the node j will be broken, inducing further redistribution of flow of $F_j + \Delta F_j$ and potentially further breakdown of nodes. After the cascading failure process stops, we calculate the avalanche size S_i, which is defined as the total number of broken nodes through the process induced by attacking the node i initially.

The LWFRR can be explained by taking the scenario of information traffic on the Internet as an example. After congestion or breakdown occurs in a router, information flow is rerouted to bypass it, which evidently leads to flow increase in other routers. Since, in general, a node of higher traffic flows always has a stronger ability to handle traffic transmission, i.e. a node's threshold is proportional to its weight, it is reasonable to preferentially reroute traffic along those higher-capacity nodes to maintain normal functioning of traffic and try to avoid further congestions. For simplicity, we assume that the additional flow received by a router is proportional to its weight. When a node receives extra flow, its total flow may exceed its capacity, with packets built up in its buffer, and congestion occurs consequently. The same story could happen again for those neighbors of the newly broken nodes. As a result, an avalanche of overloads emerges on the network. Another related example for LWFRR exists in our daily public traffic system. If we map each road as a node, and the intersection between two roads as a link between them, then we can get a traffic network [28]. Once a road is congested because of some traffic accidents, the congestion of the road will immediately increase the burden of adjacent roads. It is natural that those roads with more traffic lanes would share more additional traffic. At last, we want to remark that our LWFRR may be more relevant to

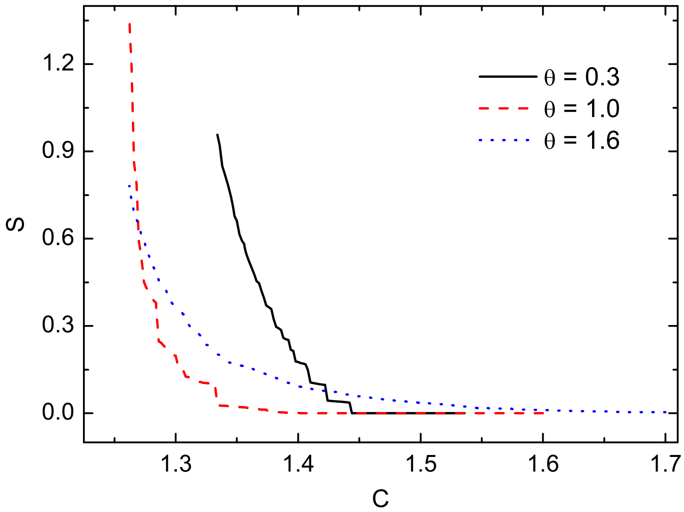

Figure 2. Average avalanche size S as a function of tolerance parameter C for several values of θ on BA scale-free networks with total size $N = 5000$ and average connectivity $\langle k \rangle = 4$. The cascading failure threshold $\mathcal{C}$ beyond which broken nodes' growth occurs ($S > 0$) is smaller for $\theta = 1.0$ as compared to the cases of $\theta = 0.3$ and 1.6. Each curve is obtained by averaging over experiments on 20 independent networks.

the dynamics in the protein interaction network or cellular networks. In such systems, when a certain element, such as a protein or a substrate fails or is removed, others should take over its burden to survive the lack thereof.

3. Results and discussion

To explore the effect of a small initial attack on our cascading model, we defunctionalize only one node initially and calculate S_i, which is the avalanche size (the total number of broken nodes) induced by defunctionalizing i, after the cascading process is over. To quantify the robustness of the network, we adopt the average avalanche size $S = \sum_i S_i/N$, obtained via summation over all the avalanche sizes by defunctionalizing each node initially at each time divided by the total number of nodes N. Given a value of θ, when the value of C is sufficiently small, we can imagine that it is easy for the whole network to fully collapse in the case of an arbitrary node's failure. On the other hand, for sufficiently large C, the destructive impacts caused by the failure of those nodes with small degrees could be absorbed by nodes with larger ones, and no cascading phenomenon emerges. Thus, with the increase of C, there should be some crossover behavior of the system from large scale breakdown to no breakdown, going through small scale ones. Thus, S can be regarded as an order parameter to characterize the robustness of the network.

Figure 2 shows S as a function of the tolerance parameter C for several values of θ for BA networks with size $N = 5000$ and average connectivity 4. Each curve is obtained by averaging over experiments on 20 independent networks. For each curve, a crossover behavior occurs at a critical threshold of C. When the value of C is beyond this threshold, no cascading failure arises and the system maintains its normal and efficient functioning; while for the case of C smaller than the threshold, S increases rapidly from zero and

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cascading failure emerges, causing the whole or part of the network to stop working. Hence the threshold is the least value of protection strength to avoid cascading failure. Apparently, the lower the value of it, the stronger the robustness of the network against cascading failure.

From figure 2, we note that it is the case for $\theta = 1.0$ that a cascading process occurs at the latest with the decrease of C, which is in accordance with the edge-based cascading model [24]. In order to understand this observed universal phenomenon, we provide some theoretical analysis. To avoid the emergence of cascading failure, the following condition should be satisfied:

$$\frac{F_i w_j}{\sum_{l \in \Omega_i} w_l} + F_j < \mathcal{F}_j. \tag{3}$$

Rewriting the arguments as a function of the node degree, we have

$$\frac{k_i^\theta k_j^\theta}{\sum_{l \in \Omega_i} k_l^\theta} + k_j^\theta < C k_j^\theta. \tag{4}$$

The denominator of the first term of the above inequality can be written as

$$\sum_{l \in \Omega_i} k_l^\theta = \sum_{k'=k_{\min}}^{k_{\max}} k_i P(k'|k_i) {k'}^\theta, \tag{5}$$

where $P(k'|k_i)$ is the conditional probability that a node of k_i has a neighbor of k'. Since BA networks have no degree–degree correlation [2], we have $P(k'|k_i) = k'P(k')/\langle k \rangle$. Combining (4) and (5) yields

$$\frac{k_i^{\theta-1}\langle k \rangle}{\langle k^{\theta+1} \rangle} + 1 < C. \tag{6}$$

From the above inequality, the critical tolerance parameter, denoted by $\mathcal{C}$, can be calculated by considering the ranges of $\theta < 1$, $\theta = 1$, and $\theta > 1$, respectively:

$$\mathcal{C} = \begin{cases} k_{\max}^{\theta-1}\langle k \rangle / \langle k^{\theta+1} \rangle + 1, & \theta > 1, \\ \langle k \rangle / \langle k^2 \rangle + 1, & \theta = 1, \\ k_{\min}^{\theta-1}\langle k \rangle / \langle k^{\theta+1} \rangle + 1, & \theta < 1, \end{cases} \tag{7}$$

where $k_{\max}$ and $k_{\min}$ are the maximum and minimum node degrees of the network. To find the rank of these critical values, we first consider the ratio of $\mathcal{C}(\theta > 1) - 1$ to $\mathcal{C}(\theta = 1) - 1$:

$$\frac{\mathcal{C}(\theta > 1) - 1}{\mathcal{C}(\theta = 1) - 1} = \frac{k_{\max}^{\theta-1}\langle k^2 \rangle}{\langle k^{\theta+1} \rangle}. \tag{8}$$

Since the degree distribution of BA networks in the large limit size N is $P(k) = 2k_{\min}^2 k^{-3}$ [1], we have $k_{\max} = k_{\min}^2 \ln N$, $\langle k^2 \rangle = \int_{k_{\min}}^{k_{\max}} k^2 P(k)\,\mathrm{d}k = k_{\min}^2 \ln N$ and $\langle k^{\theta+1} \rangle = \int_{k_{\min}}^{k_{\max}} k^{\theta+1} P(k)\,\mathrm{d}k = (2/(\theta-1))k_{\min}^{\theta+1}(N^{(\theta-1)/2} - 1)$. Substituting these expressions into equation (8), we obtain

$$\frac{\mathcal{C}(\theta > 1) - 1}{\mathcal{C}(\theta = 1) - 1} = \frac{(\theta-1)N^{(\theta-1)/2}\ln N}{2(N^{(\theta-1)/2} - 1)}. \tag{9}$$

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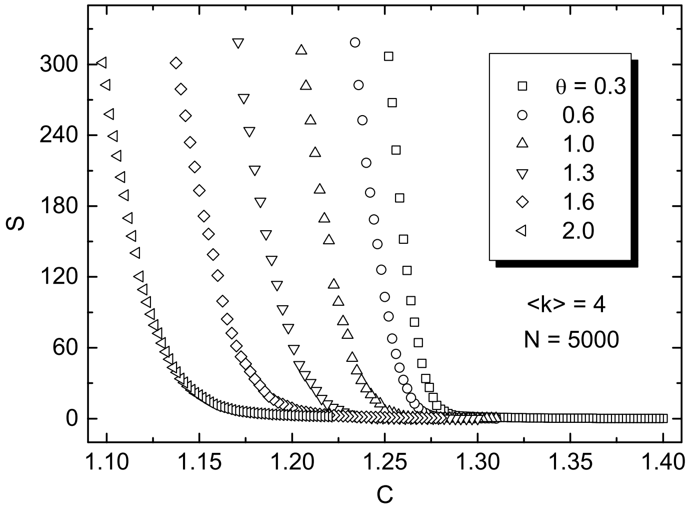

Figure 3. Average avalanche size S as a function of the tolerance parameter C on BA scale-free networks with size $N = 5000$. The average degree of the networks is fixed as $\langle k \rangle = 4$ and the value of the weight parameter θ is varied as 0.3, 0.6, 1.0, 1.3, 1.6 and 2.0, from right to left. The data are obtained by averaging over experiments on 20 independent networks.

In the large limit of N, both the denominator and numerator approach infinity, equation (9) can be calculated by differentiating the denominator and numerator with respect to N, respectively, which yields $1 + \ln N^{(\theta-1)/2}$. For $\theta > 1$, equation (9) is larger than 1, which indicates $\mathcal{C}(\theta > 1) > \mathcal{C}(\theta = 1)$. Similarly, we can get $\mathcal{C}(\theta < 1) > \mathcal{C}(\theta = 1)$. Thus, in the case of $\theta = 1$, the threshold of the tolerance parameter below which the cascading failure emerges is the smallest one compared to that in other cases of θ, confirming our simulation results in figure 2. Here, we want to point out that there is some discrepancy between the theoretical estimations of the critical tolerance parameter and those obtained by computer simulation, which results from one approximation in the analytical treatment and also the finite size of the networks. This result might give us some insights in designing networked system in practice (for example, communication networks): if we hope the designed system shows robust behavior against the emergence of cascading failure, we should better correlate the node's weight linearly with its degree, given that each broken node releases its load according to the LWFRR.

So far, we have shown by both simulation and analysis that, when the weight parameter $\theta = 1.0$, the threshold of the tolerance parameter is minimal for the emergence of cascading failure on BA scale-free networks. But it is not the whole story. Generally, besides the onset of the avalanche, we also care about how the avalanche size changes depending on the tolerance parameter. To this end, we also investigated the evolution of the average avalanche size on the network after the emergence of cascading failure by further decreasing C while keeping the weight parameter θ fixed. The simulation results are presented in figure 3. It is found that, on networks with larger θ, the average avalanche size develops at a slower rate compared with those cases of smaller θ. This point is reflected by the smaller avalanche size for larger values of θ at a given tolerance C. The phenomenon can be understood as follows. For heterogeneous networks with a large value of weight parameter θ, if the nodes with large degrees are initially attacked, due to their possessing high capacity, it is easier for them to induce further breakdown of other nodes, especially those with smaller degrees. This gives rise to the result that cascading failure is easier to occur for large θ, as was shown in figure 2. On the other hand, for larger θ, if it was nodes with smaller degrees that were attacked initially, the additional flow they released to their neighbors would be more easily absorbed by those neighbors with large degrees due to their higher tolerance, possibly without inducing further cascading failures. Moreover, it is known that, in BA scale-free networks, the nodes with large degrees only occupy a very small proportion and most nodes are those with small ones. Note that the average avalanche size S is obtained by averaging over S_i of all nodes in the network. Combining these elements, we can conclude that the networks with larger weight parameters are prone to cascading failure caused by a single node's breakdown, whereas, after cascading failures emerged, their magnitudes develop more slowly compared to the cases with smaller weight parameters. As for θ smaller than unity, since the capacities of high-degree nodes are considerably inhibited, the high-degree nodes are more easily broken, induced by the failures of their small degree neighbors, i.e. the damage from small degree neighbors is more difficult to be absorbed by high-degree nodes (for example, in the case of $\theta = 0.1$, the absolute difference of the weight between a node with degree 4 and a node with degree 100 is about 0.436, while for $\theta = 1.0$ the difference is 96). Hence, the avalanche size grows more rapidly with the decrease of C than in the case of $\theta = 1$, as reflected in figure 3.

Now let us investigate the avalanche size distribution $P(S)$, which is a conventional parameter to characterize the avalanche dynamics. To explore the statistical features of the avalanche sizes, we continuously increase the flow along nodes by a small constant δ, starting from a load-empty network. This method is essentially the same as the sandpile model and fiber bundle model [6, 15, 16]. For simplicity, the threshold of each node is assigned to be its weight. At any time step, we add δ to the flow of one randomly selected node. With time increasing, the flow accumulated on the network increases continuously. If the flow of a node exceeds its prescribed threshold, the node is broken and the flow on it will be redistributed to its neighboring nodes according to the LWFRR, which would induce other potential breakdown of the nodes. A sampling time step is counted after all the cascading events (if any) are over. At each time step, we record the number of broken nodes as the avalanche size at this time step. Then we recover all broken nodes and set their flows to zero. At the next time step, we add flow δ to a randomly selected node, as before, and repeat the cascading process. The total sampling time is up to 10^8. After that, we obtain the avalanche distribution of the sizes recorded at each time step. If no cascading failures occur, S is zero.

We have explored the avalanche size distribution $P(S)$ on several BA scale-free networks with $N = 5000$ and $\langle k \rangle = 4$. The parameter θ is varied in the range $0.1 \leq \theta \leq 1.8$. The simulation results are shown in figures 4 and 5. As can be seen from these two figures, the form of the calculated $P(S)$ depends strongly on the value of θ, but slightly on δ (see figure 5). Here we want to remark that the selection of the value of δ should not be too large, as was always done in the sandpile model and the fiber bundle model [6], [14]–[16]. For smaller θ, $P(s)$ shows an exponential decay behavior decreasing rapidly at larger magnitudes of S, which implies large avalanche events are scarce (figure 4(a)). Such behavior of $P(S)$ is often called 'subcritical'. For the moderate θ, e.g. $\theta = 0.9, 1.0$, the curves lie on a straight line fairly well (figure 4(b)), apparently satisfying the power-law decay form, which is a prominent phenomenon empirically

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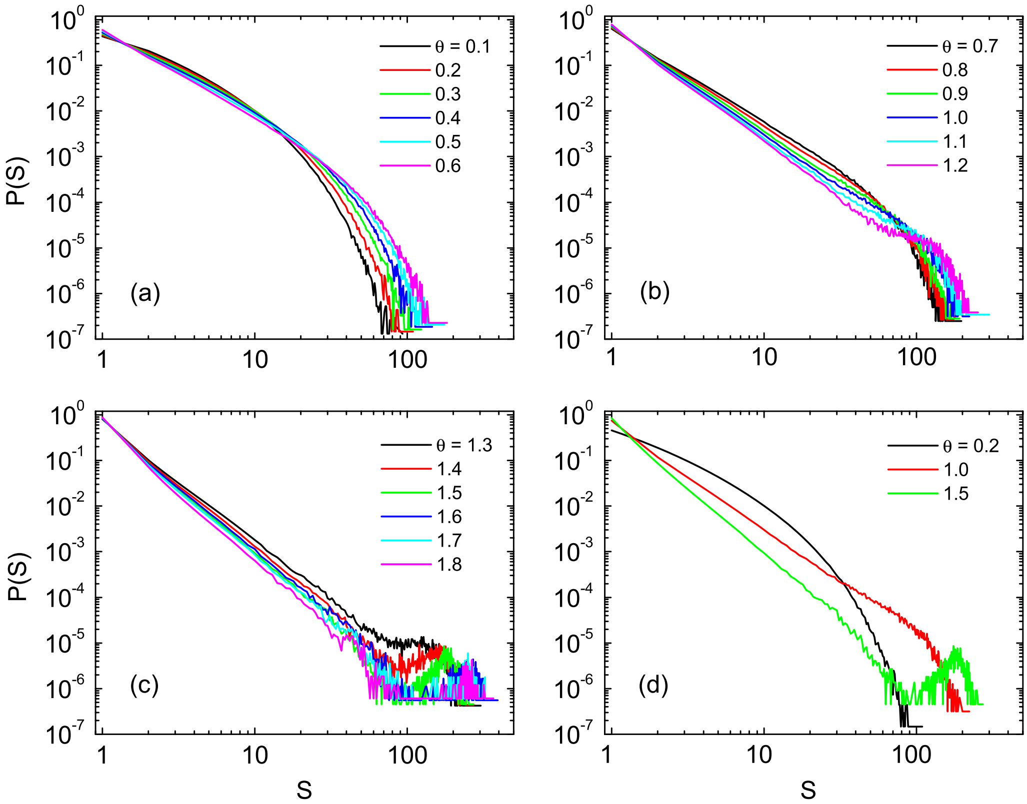

Figure 4. Distributions of the avalanche size from time step $t = 1$ to 10^8 for different values of θ on BA networks with size $N = 5000$ and average degree $\langle k \rangle = 4$. (a) represents $P(s)$ for smaller values of the weight parameter $0.1 \leq \theta \leq 0.6$; (b) represents $P(s)$ for moderate values of the weight parameter $0.7 \leq \theta \leq 1.2$; and (c) represents larger values of the weight parameter $1.3 \leq \theta \leq 1.8$. (a), (b) and (c) correspond to the regions of 'subcritical', 'coexistence of subcritical–critical–supercritical' and 'supercritical', respectively. For clarification, three typical curves in each region are shown in panel (d), corresponding to $\theta = 0.2, 1.0, 1.5$, respectively. Other parameters: $N = 5000$, $\delta = 1.0$.

observed in previously studied cascading failure processes and the mark of the emergence of 'criticality' [7, 11]. The truncation at the tails are due to a finite size effect. As shown in figure 4(c), for large values of θ, $P(S)$ exhibits a pronounced peak structure for larger cascading events, while the power-law feature still remains for smaller magnitudes. Accordingly, such behavior of $P(S)$ is often called 'supercritical', since $P(S)$ increases at larger magnitudes (though it eventually falls off at even larger magnitudes), which means that a small initial perturbation can disturb a very large area of the whole system. The above phenomena can be understood in the following way.

On the one hand, when θ is very small (e.g. $\theta = 0.2$), the tolerance thresholds of the nodes are comparably small. In this case, it would be more frequent for the emergence of overloading of the nodes (but with *small* additional flow to be redistributed) when continuously adding load to the network. On the other hand, the BA scale-free networks have a slightly disassortative degree–degree correlation [29] (also explained in

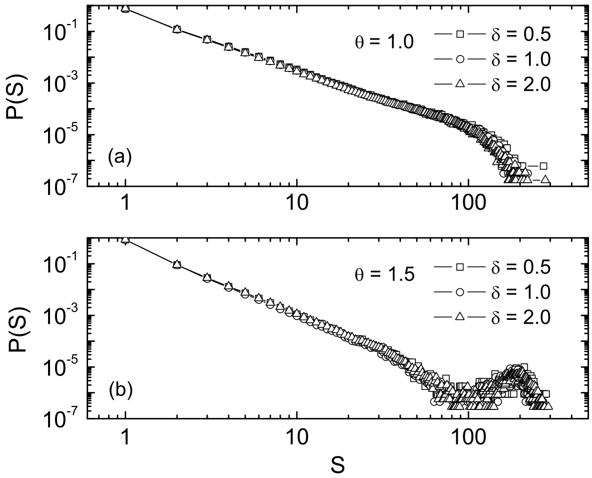

Figure 5. The avalanche size distribution for two different values of θ and several values of δ. (a) is for $\theta = 1.0$ and (b) for $\theta = 1.5$. The size of the network is $N = 10^5$ and the total sampling time is 10^8. As can be seen, the form of the curves depends strongly on θ, but only slightly on δ considered here.

the next paragraph), i.e. the nodes with large (small) degrees have somewhat greater probability to attach to other nodes with small (large) degrees. This configuration of connections among the nodes would suppress the scale of the cascading failure since the nodes with large degrees could dilute considerably the destructive impact caused by those nodes with small tolerance (remember the disassortative property of the BA networks). With the increase of θ, the ability of the nodes to bear additional load is gradually promoted. The continuously adding load may not cause any cascading for a *long* time. In this sense, the network can self-organize to a state where local cascading failure events are relatively scarce compared to the case of small θ but, once a cascading failure occurs, it may affect a reasonably large area of the network, i.e. cascading events with large magnitudes become possible. For even larger θ (e.g. $\theta = 1.5$), the avalanche size caused by a single failure can develop to a more significant large scale; hence the 'supercritical' spreading behavior of the failures. Here, we want to remark that, with the increase of θ, both the power-law decay behavior of $P(S)$ and the pronounced peak where increasing $P(S)$ emerges smoothly, and the magnitude of the peak value also increases gradually (from figures 4(a)–(c). For clarification, we show three typical curves in each region in figure 4(d), corresponding to $\theta = 0.2, 1.0, 1.5$, respectively). Thus, the crossover from the 'subcritical' to 'critical' to 'supercritical' behavior with increasing θ is of a continuous or 'second-order' nature. At the present time, unfortunately, it is hard for us to give an analytical expression for $P(S)$, since the cascading process considered here is not a type of conventional branching process and cannot be addressed by following the method proposed in [25].

Motivated by the above analysis, we would like to investigate in detail how the configuration of the connections among nodes impacts the cascading failure spreading. It has been pointed out that the existence of degree correlations among nodes is an important property of real networks [29, 30]. In a network with degree correlation, there exist certain relationships between network nodes. The degree correlations are often

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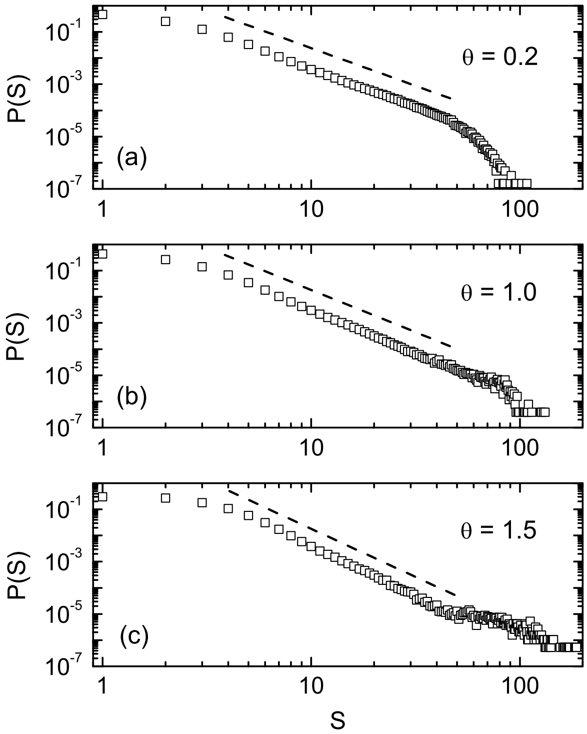

Figure 6. The avalanche size distribution for three different values of θ on weighted assortative scale-free networks. (a), (b) and (c) correspond to $\theta = 0.2$, 1.0 and 1.5, respectively. $N = 5000$ and $\delta = 1.0$. The Pearson correlation coefficient of the networks is $r \approx 0.19$. Simulation data are obtained by an average over 20 experiments on several networks. The dashed lines are guides to the eye.

named, respectively, 'assortative mixing', i.e. a preference for high-degree nodes to attach to other high-degree nodes, and 'disassortative mixing', i.e. high-degree nodes attach to low-degree ones [29, 30]. We first generate degree-correlated networks from the BA networks by means of the reshuffling method proposed in [31]. Starting from a BA network, at each step two edges of the network are chosen at random. The four nodes attached to the two edges are ordered with respect to their degrees. Then the edges are rewired in such a way that one edge connects the two nodes with the smaller degrees and the other connects the two nodes with the larger degrees; otherwise, the edges are randomly rewired. In the case when one or both of these new edges already existed in the network, the step is discarded and a pair of other edges is selected. A repeated application of the rewiring step leads to an assortative network. For producing disassortative networks, we change the way of building new edges used in the above reshuffling method into that the node of the largest degree connects to the nodes of the smallest degree and the two other nodes are connected. It is worth noting that the algorithm does not change the degree distribution in the given network [31].

The extent of assortative/disassortative mixing of a network is usually characterized by the so-called Pearson correlation coefficient r [29], where $|r| < 1$. The larger (smaller) the value of $r > 0$ ($r < 0$), the more assortative (disassortative) the generated network is. In figures 6 and 7, we summarize our simulation results for the distribution of the

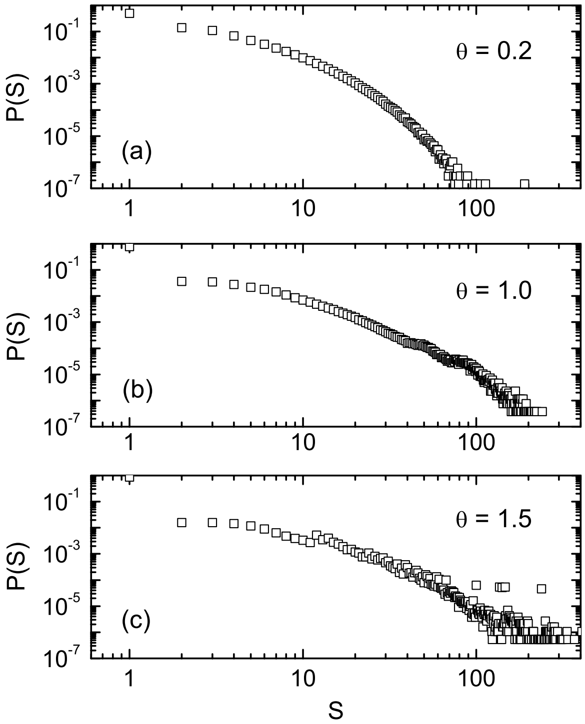

Figure 7. The avalanche size distribution for three different values of θ on weighted disassortative scale-free networks. (a), (b) and (c) correspond to $\theta = 0.2$, 1.0 and 1.5, respectively. $N = 5000$ and $\delta = 1.0$. The Pearson correlation coefficient of the networks is $r \approx -0.135$. Simulation data are obtained by an average over 20 experiments on several networks.

avalanche sizes on both assortative and disassortative scale-free networks, respectively. The Pearson correlation coefficient r is about 0.19 for assortative networks and -0.135 for disassortative ones. We have checked that the connectivity of the generated networks holds for these two values of r. For assortative scale-free networks, we can see from figure 6 that, for all values $\theta = 0.2, 1.0$ and 1.5 considered here, the distribution of the avalanche sizes displays power-law decay behavior for intermediate cascading events, which means the emergence of critical spreading behavior. With the increase of θ, cascading events with larger magnitudes occur with greater chance, in accord with the results discussed before. For dissassortative scale-free networks, however, $P(s)$ more likely displays an exponential decay form rather than a power-law one, as shown in figure 7. This implies that the dissassortative correlation among the nodes indeed inhibits the emergence of a large area of avalanches on the network, as was found above for the BA networks (as mentioned above, the BA networks possess very slightly disassortative mixing).

As to why assortative networks favor the emergence of 'criticality', we argue that the similarity among directly connected nodes contributes to this dynamical phenomenon. In assortative networks, nodes with similar degrees are more likely to be connected, which leads to the tolerances between pairs of connected nodes also being similar. By continuously adding load to randomly selected nodes, once a failure arises, it may induce large scale successive breakdown of the nodes, hence the failures can spread far away (because of the lack of nodes with *higher* tolerance, due to their similarity, to efficiently absorb the damage). In this sense, the cascading model is more like a sandpile model where the toppling threshold for all nodes is fixed as a constant [6]. Combining the results in figures 4, 6 and 7, we can conclude that the existence of degree correlation among the nodes is an important factor to affect the dynamical behavior of the cascading failure spreading on scale-free networks. In particular, it is easier for a single failure on assortative scale-free networks to induce a large scale cascading event compared to the cases of no correlation and disassortative networks.

4. Conclusion

To sum up, we have studied the cascading reaction behaviors on BA scale-free networks with respect to small node initial attacks. Each node is endowed with a weight k^θ describing its ability of handling load, and a tolerance Ck^θ when confronting additional flow. We provide theoretical estimations of the critical tolerance parameter value for the onset of the cascading failure. Though, with the decrease of the tolerance, the networks with large values of θ are prone to the emergence of avalanches triggered by a randomly attacked node, the magnitude of the avalanche size on them grows more slowly compared to that on those networks with small θ. Depending on the weight parameter θ, subcritical, critical and supercritical avalanche dynamics may occur. For large θ, cascading failure can spread over a large portion of the network. In other words, the smaller the value of θ, the smaller the region the cascading failure is confined to. We have also studied the effect of the degree correlation among nodes on the cascading failure spreading on weighted scale-free networks. It was found that single failure on assortative scale-free networks may be more likely to induce a series of subsequent failure events up to a large scale, and the cascading dynamics may easily self-organize to a 'criticality' state on such assortative scale-free networks. These results indicate the significant roles of weights and degree correlation among nodes on dynamical processes taking place on complex networks. The presented model may help us understand cascading phenomena in the real world and may shed light on designing a control strategy to prevent various cascading-failure-induced disasters.

Acknowledgments

We thank our colleague Professor Kai-Hau Yeung for useful discussions. Much help from Dr Zhihai Rong in the preparation of this paper is greatly appreciated.

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J. Stat. Mech. (2008) P05013