A LETTERS JOURNAL EXPLORING
THE FRONTIERS OF PHYSICS

OFFPRINT

Universal dynamics on complex networks

WEN-XU WANG, LIANG HUANG and YING-CHENG LAI

EPL, **87** (2009) 18006

A Letters Journal Exploring the Frontiers of Physics
July 2009
EPL, **87** (2009) 18006 www.epljournal.org
doi: 10.1209/0295-5075/87/18006

Universal dynamics on complex networks

Wen-Xu Wang[1], Liang Huang[1] and Ying-Cheng Lai[1,2]

[1] *Department of Electrical Engineering, Arizona State University - Tempe, AZ 85287, USA*
[2] *Department of Physics and Astronomy, Arizona State University - Tempe, AZ 85287, USA*

received 7 April 2009; accepted in final form 22 May 2009
published online 17 July 2009

PACS 89.75.Hc – Networks and genealogical trees
PACS 89.20.Hh – World Wide Web, Internet
PACS 05.10.-a – Computational methods in statistical physics and nonlinear dynamics

Abstract – We uncover a class of universal dynamics on weighted complex networks. In particular, we find that by incorporating a universal weighting scheme into real-world networks, the topological details of various real-world networks, whether biological, physical, technological, or social, have little influence on typical dynamical processes such as synchronization, epidemic spreading, and percolation. This striking finding is demonstrated using a large number of real-world networks and substantiated by analytic considerations. These findings make possible generic and robust control strategies for a variety of dynamical processes on complex networks.

Universality is one of the most fundamental issues in physics. Critical phenomena and universal scaling laws associated with phase transitions in a large variety of non-equilibrial physical and chemical systems [1,2] and universal routes to chaos in nonlinear dynamical systems [3] are classical examples. Searching for universality is thus one of the most pursued endeavors, particularly in statistical and nonlinear physics. In the past decade there has been a tremendous amount of interest in complex networks, as stimulated by the discoveries of the small-world [4] and the scale-free topologies [5]. Various dynamical processes on complex networks such as synchronization [6], propagation [7], and transportation [8], have also been investigated. A question is then whether there exist *universal dynamics* on complex networks. In particular, given networks from different contexts, is there a universal class of dynamics that absolutely has no dependence on structural details of the network? Here we provide a surprising but an affirmative answer to the above question. In particular, we find the existence of weighting schemes for which the details of various real-world networks, whether biological, technological, or social, have little influence on typical dynamical processes such as synchronization, epidemic spreading, and percolation. Here, in our computation, we use the topologies of a number of real-world networks from different disciplines and impose a controllable weighting schemes to model the coupling configuration of the network. In other words, by incorporating our proposed weighting scheme into any complex networks, the networks exhibit universal dynamics, regardless of their difference in topology. This striking universality in network dynamics is demonstrated by using a large number of real-world networks and substantiated by analytic considerations. The universality makes possible generic and robust control strategies for a variety of dynamical processes on networks arising from different contexts.

The key to our success in searching for universal network dynamics lies in considering weights on networks. Indeed, in real-world networks, interactions among nodes are not uniform but typically are heterogeneous, or weighted. To be general, we shall examine both symmetric and asymmetric weighting schemes. To be able to carry out concrete and quantitative analysis to cover as many types of network dynamics as possible, we choose to examine the behavior of the largest eigenvalue of the weighted adjacency matrix, denoted by λ_N. The role of λ_N in different types of dynamics can be appreciated through the following examples: in a heterogeneous dynamical network, λ_N determines the emergence of coherence [9]; in epidemic spreading, λ_N sets the infection threshold for outbreak of virus and shapes the onset of percolation transition [7,10]; in general, λ_N governs the linear stability of system of coupled dynamical elements [11,12]. Our approach is to explore the dependence of λ_N on some parameters, say α, that characterizes the weighting scheme for real-world networks. We shall present results with twelve different realistic networks ranging from the neural network of *C. Elegans* in biology to the Internet at the level of autonomous systems and to social networks such

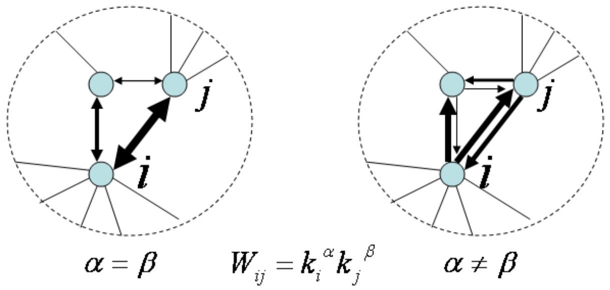

Fig. 1: (Colour on-line) Weight of edge ij is determined by the degrees of nodes i and j: $W_{ij}=k_i^{\alpha}k_j^{\beta}$. For $\alpha=\beta$, the weighting scheme is symmetric, which has been observed for realistic networks [13]. For $\alpha \neq \beta$, the interactions are asymmetric and directed: $W_{ij} \neq W_{ji}$ for $k_i \neq k_j$.

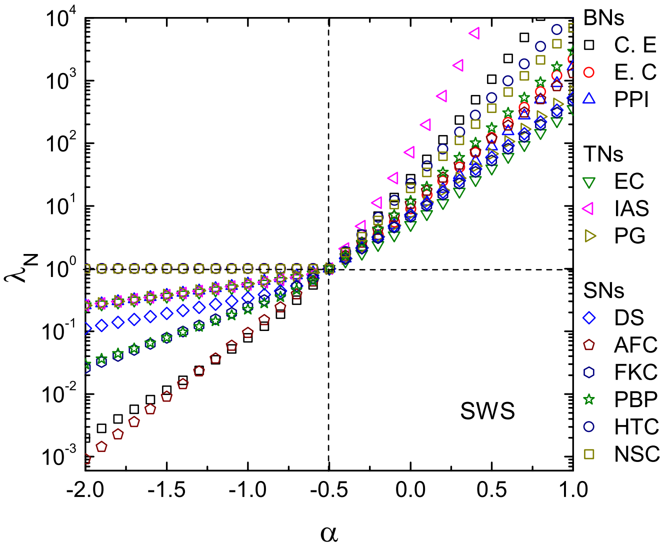

Fig. 2: (Colour on-line) For the symmetric weighting scheme, the largest eigenvalue λ_N of the weighted adjacency matrix as a function of the weighting parameter α for twelve different real-world networks. They are: 1) the neural network of *C. Elegans* (denoted by C. E) [4], 2) the transcriptional regulation network of *E. coli* (E. C) [14], 3) the protein-protein interaction network of yeast (PPI) [15], 4) the electronic circuit network (EC) [16], 5) the Internet at the level of autonomous systems (IAS) [17], 6) the Western States Power Grid of the United States (PG) [4], 7) the dolphin social network (DS) [18], 8) the network of American football games among colleges (AFC) [19], 9) the social network of friendships of a karate club (FKC) [20], 10) the network of political book purchases (PBP) [21], 11) the high-energy theory collaboration network (HTC) [22], and 12) the collaboration network of scientists working on network theory and experiment (NSC) [23]. Examples 1)–3) belong to biological networks, 4)–6) are physical and technological networks, and 7)–12) are social networks. Dashed lines indicate the place where universal critical dynamics arise: $\alpha_c=-0.5$ and $\lambda_N(\alpha_c)=1$.

as the American football games. See fig. 2 for details. For any two different types of networks, we expect the λ_N-α curves to be distinct and generically to intersect at some value, say α_c. The striking finding is that all the λ_N-α curves from the twelve completely different networks intersect exactly at the same α_c! The critical values of α_c and λ_N at the intersection point depend only on the weighting scheme and they do not depend on the topological details of the network. This means that, at the intersection point, the specific structural details of different networks disappear and the network dynamics become universal. To place our finding on a firm ground, we develop an analytic theory for determining the critical values, with predictions agreed well by results from real-world examples. To provide direct evidence for universal dynamics with respect to actual dynamical processes, we present results from transition to synchronization in the Kuramoto type of phase coupled dynamics on weighted scale-free networks.

We use a generic weighting schemes to search for universal dynamics. As illustrated in fig. 1, the weight between nodes j and i is $W_{ij}=A_{ij}k_i^{\alpha}k_j^{\beta}$, where k_i and k_j are the degrees of i and j, respectively, α and β are control parameters, and A is the unweighted adjacency matrix of the network defined by $A_{ij}=1$ if nodes i and j are connected and $A_{ij}=0$ otherwise. The coupling strength can be both symmetric and asymmetric. The asymmetric scheme takes into account the fact that influences from a node to its neighbors are usually the same, but influences from different nodes can be different. In realistic situations, W characterizes, for example, couplings among oscillators, infection probabilities and edge removal probabilities, etc.

We first present evidence of universal dynamics for a large number of real-world networks in terms of the dependence of λ_N on the weighting parameters α and β, as shown in fig. 2 for the symmetric weighting scheme $\alpha=\beta$. The topology is taken from the real-world networks, but the weights of links are modelled by our weighting scheme. We observe that the relations between λ_N and α $(=\beta)$ share the same intersecting point: $\alpha_c=-0.5$ for which $\lambda_N(\alpha_c)=1$. Note that, these networks differ significantly from each other in terms of structural properties such as degree distribution, clustering coefficient, average network distance, degree-degree correlation, and network size etc. However, for $\alpha=\beta=\alpha_c$, typical dynamical processes as determined by λ_N are universal, regardless of any network details. That is, distinct real-world networks will support identical dynamics such as synchronization and epidemic spreading. For an asymmetric weighting scheme $\beta=0$, a similar phenomenon has been observed, except that the critical point is now at $\alpha_c=-1$, as shown in fig. 3, demonstrating that universal dynamics can also occur in networks with directed interactions.

We now provide an analytic theory to explain the occurrence of the universal dynamics. Elements of the weighted adjacency matrix W are $W_{ij}=A_{ij}k_i^{\alpha}k_j^{\beta}$. The largest eigenvalue λ_N can be written as $\lambda_N=e_N^T W e_N$, where e_N is the eigenvector associated with λ_N. If $|\lambda_N|$ is significantly larger than $|\lambda_{N-1}|$, we have $e_{N,i}=cs_i$,

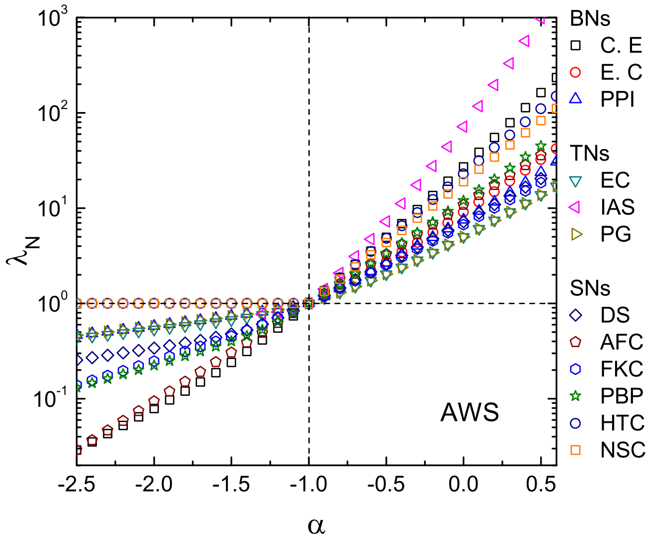

Fig. 3: (Colour on-line) For the asymmetric weighting scheme, λ_N as a function of α for the same twelve real-world networks in fig. 2. The occurrence of a universal cross point persists, except that it occurs now at $\alpha_c = -1$.

where c is a constant and s_i is the summation over row i, *i.e.*, $s_i = \sum_{j=1}^{N} W_{ij}$. We have $\lambda_N = c^2 \sum_i \sum_j W_{ij} s_i s_j = c^2 \sum_i \sum_j A_{ij} k_i^{\alpha} k_j^{\beta} s_i s_j$. For a given node i, we have $s_i = \sum_{l=1}^{N} A_{il} k_i^{\alpha} k_l^{\beta} = k_i^{\alpha+1} \sum_{k'=k_{\min}}^{k_{\max}} P(k'|k_i) k'^{\beta}$, where $P(k'|k_i)$ is the conditional probability that a node of degree k_i has a neighbor of degree k'. Neglecting the degree-degree correlation among nodes yields $P(k'|k_i) = k' P(k')/\langle k \rangle$, where $P(k')$ is the degree distribution of the network and $\langle k \rangle$ is the average node degree. We thus obtain

$$s_i = k_i^{\alpha+1} \sum_{k'=k_{\min}}^{k_{\max}} \frac{k'^{\beta+1} P(k')}{\langle k \rangle} = \frac{k_i^{\alpha+1} \langle k^{\beta+1} \rangle}{\langle k \rangle}, \qquad (1)$$

where the identity $\sum_{k'=k_{\min}}^{k_{\max}} k'^{\beta+1} P(k') = \langle k^{\beta+1} \rangle$ has been used. We can then write

$$\lambda_N = \frac{c^2 \langle k^{\beta+1} \rangle^2}{\langle k \rangle^2} \sum_i k_i^{2\alpha+1} \sum_j A_{ij} k_j^{\alpha+\beta+1} = \frac{c^2 N \langle k^{\beta+1} \rangle^2 \langle k^{\alpha+\beta+2} \rangle \langle k^{2\alpha+2} \rangle}{\langle k \rangle^3}. \qquad (2)$$

The coefficient c is given by $1 = \sum e_{N,i}^2 = c^2 \sum_{i=1}^{N} s_i^2$, where s_i is given by eq. (1). This yields

$$c^2 = \frac{\langle k \rangle^2}{N \langle k^{\beta+1} \rangle^2 \langle k^{2\alpha+2} \rangle}. \qquad (3)$$

Inserting this into λ_N, we obtain

$$\lambda_N = \frac{\langle k^{\alpha+\beta+2} \rangle}{\langle k \rangle}. \qquad (4)$$

We see that, for the critical symmetric $[\alpha_c (= \beta_c) = -0.5]$ and asymmetric ($\alpha_c = -1$ for $\beta = 0$) weighting schemes,

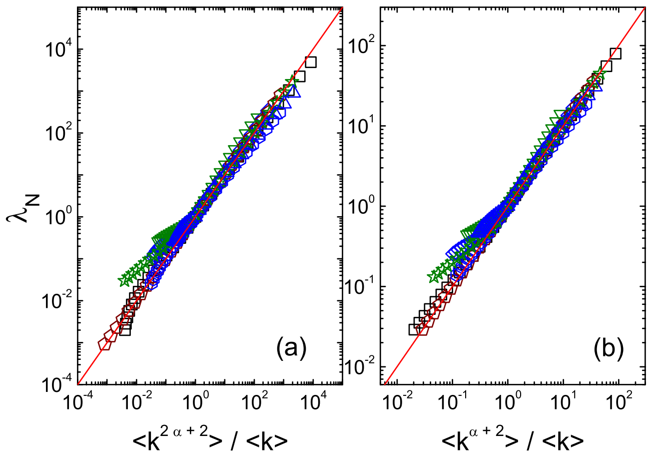

Fig. 4: (Colour on-line) Numerical verification of the analytical predictions λ_N for (a) symmetric ($\alpha = \beta$) and (b) asymmetric ($\beta = 0$) weighting schemes *vs.* the respective rescaled quantity by eq. (4) for all twelve real-world networks. The solid lines have the unit slope.

λ_N no longer depends on the network details: $\lambda_N = 1$, signifying universal critical dynamics.

The above analytical estimations require only the local topological information of nodes and the weighting schemes. Numerical results explicitly verifying eq. (4) are shown in figs. 4(a) and (b), respectively. There is an excellent agreement between the predictions and numerical results with real-world networks. In particular, on the plot of λ_N *vs.* some rescaled quantity ($\langle k^{2\alpha+2} \rangle / \langle k \rangle$ for the symmetric-coupling case and $\langle k^{\alpha+2} \rangle / \langle k \rangle$ for the asymmetric case), results from real networks that we have examined collapse onto a straight line of slope one. We note that, for $\alpha = \beta = 0$, λ_N reduces to $\langle k^2 \rangle / \langle k \rangle$, which is the inverse of the threshold for epidemic outbreak and the critical transition point to coherence in coupled nonidentical oscillators under the mean-field approximation in unweighted networks [7,9]. For the percolation dynamics, the transition point occurs at the inverse of $1 + \langle k^2 \rangle / \langle k \rangle$ [24]. The reduction of our result to established results for the unweighted cases further demonstrates the validity of our theory.

Equation (4) yields a class of weighted networks with universal dynamics that occur for

$$\alpha + \beta = -1. \qquad (5)$$

Although eq. (5) is derived from general formulas for $\lambda_N(\alpha)$, the existence of the universal class of weighted networks can be predicted without referring to any network details. The weighted adjacency matrix can be denoted as $W = K^{\alpha} A K^{\beta}$, where $K_{ij} = \delta_{ij} k_i$ and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Let $V = c\tilde{k}^{-\beta}$, where c is a normalization constant and $(\tilde{k}^{-\beta})_i = k_i^{-\beta}$. We have $W \cdot V = K^{\alpha} A K^{\beta} c \tilde{k}^{-\beta} = K^{\alpha} A c \mathbf{1} = c \tilde{k}^{1+\alpha} = c \tilde{k}^{-\beta} = 1 \cdot V$, where $\mathbf{1} = [1, 1, \ldots, 1]^T$ and the relation (5) is used. Since W is connected and all elements of W and V

are positive, V can be proved to be the eigenvector associated with the largest eigenvalue [25], and the largest eigenvalue is thus exactly 1. We can argue that all the eigenvalues of the weighted adjacency matrix W are real, regardless of the values of α and β. In particular, performing a similarity transformation on W, we obtain $W' = K^{(\beta-\alpha)/2} W K^{(\alpha-\beta)/2} = K^{(\beta-\alpha)/2} K^{\alpha} A K^{\beta} K^{(\alpha-\beta)/2} = K^{(\alpha+\beta)/2} W K^{(\alpha+\beta)/2}$. Since W' is a symmetric matrix, all its eigenvalues are real. Further, since W' is a similarity matrix to W, all the eigenvalues of W are the same as W' and are real as well.

Our analysis so far is based on the largest eigenvalue of the weighted network, which gives indirect indication for the existence of universal network dynamics. We have also obtained direct evidence for the universal dynamics with respect to specific types of dynamics. Here we present one example: transition to synchronization in phase-coupled oscillators on weighted networks modeled by (the Kuramoto paradigm [26]):

$$\dot{\theta}_i = \omega_i + \varepsilon \sum_{j=1}^{N} W_{ji} \sin(\theta_j - \theta_i), \qquad (6)$$

where θ_i and ω_i are the phase and the natural frequency of oscillator i, $N \gg 1$ is the total number of oscillators, ε is a global coupling parameter that is identical to all oscillators. A key quantity of interest is the critical coupling ε_c for the onset of synchronization. To be able to obtain analytic insights, we have chosen model scale-free networks differing in their average degrees, under symmetric or asymmetric weighting schemes. Using a standard mean-field treatment, we have obtained explicit formulas relating ε_c to the weighting parameters α and β. Figure 5 shows a typical example for $\alpha = \beta$, where the solid curves are the predicted $\varepsilon_c \sim \alpha$ relations for networks with different average degrees, and the data points are from numerical simulations. We observe again the universal point $\alpha_c = \beta_c = -0.5$, but here the quantity examined is the *actual* synchronization threshold. In addition, the dependence of ε_c on $\langle k\rangle$ shows opposite trend for the $\alpha < \alpha_c$ and $\alpha > \alpha_c$ regime. For $\alpha < \alpha_c$, networks with smaller values of $\langle k\rangle$ exhibit smaller value of ε_c and thus are more synchronizable. This is quite counterintuitive, as the results suggest that networks with more links are less synchronizable (*abnormal* synchronization regime). For $\alpha > \alpha_c$, the values of ε_c required for synchronization are smaller for larger values of $\langle k\rangle$, indicating that networks with more links are more synchronizable. This is then a *normal* synchronization regime.

From the perspective of controlling network dynamics, our results suggest the existence of some universal strategy that is applicable to networks differing in structural details. For example, we can multiply the weighted matrix W (with λ_N as the largest eigenvalues) by a control parameter η. The link weights are then $W_{ij} = \eta A_{ij} k_i^{\alpha} k_j^{\beta}$. The largest eigenvalue of the modified weighting schemes becomes $\bar{\lambda}_N = \eta\lambda_N$. At the universal point

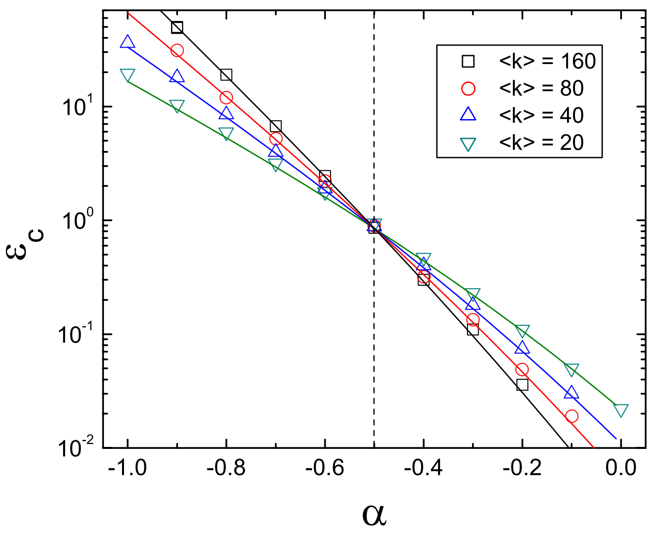

Fig. 5: (Colour on-line) For Kuramoto dynamics on symmetric coupled, weighted scale-free networks, critical global coupling strength ε_c as a function of the weighting parameter α for different values of the average degree. The network size is $N = 2000$. The occurrence of universal synchronization dynamics for $\alpha_c = -0.5$ can be seen.

we have $\bar{\lambda}_N(\alpha_c, \beta_c) = \eta$. This means that, for α and β satisfying eq. (5), tuning η can provide a general way to control diverse dynamical processes, regardless of the detailed network structures. For instance, decreasing η can inhibit the outbreak of epidemic spreading, as the threshold of the process is inversely proportional to λ_N. This is particularly important for scale-free networks due to the influences of hubs on propagations [7]. Reducing η can also favor synchronization stability of coupled linear systems [11], as well as the interconnection robustness of networks in percolation [27]. However, to enhance synchronization of nonidentical coupled oscillators, η should be increased. Imposing proper weighting schemes can then facilitate control of various network dynamics. The existence of universal critical dynamics makes generic control strategies possible that are effective for networks in different contexts.

It is noteworthy that there are dynamical processes whose main traits are not determined by the largest eigenvalue of the weighted adjacency matrix. A known example is complete synchronization of coupled identical oscillators. The stability of such a state of synchronization is measured by both the largest and the seconded smallest eigenvalues of the Laplacian matrix [28]. In fact, universality of synchronization in weighted networks has been reported, where the eigenratio can be estimated by the ratio of the largest and the smallest node intensities for a variety of weighted random networks [29].

In summary, we have discovered and established the existence of universal dynamics in weighted complex networks. For a given class of weighting scheme, there exists a group of critical points, near which the structural details of networks have little influence on various dynamical processes. The universal behavior in the collective network dynamics has important implications in significant areas of network research such as the security of complex networks. Say we wish to design a class of networks that are robust to external perturbations. Weighted networks with the weighting scheme provides a solution, as the associated network dynamics are invariant with respect to any structural changes that may be caused by attacks or random failures. Our finding can also be useful for addressing the issue of network scalability, where design principles for networks of significantly different sizes but with identical dynamics are sought.

* * *

This work was supported by AFOSR under Grant No. FA9550-07-1-0045. WXW is supported by NNSFC under Grant No. 10635040.

REFERENCES

[1] Fisher M. E., *Scaling Universality and Renormalization Group Theory* (Springer, Berlin) 1983; *Rev. Mod. Phys.*, **70** (1998) 653.

[2] Wilson K. G., *Rev. Mod. Phys.*, **55** (1983) 583.

[3] Feigenbaum M. J., *J. Stat. Phys.*, **19** (1978) 25.

[4] Watts D. J. and Strogatz S. H., *Nature*, **393** (1998) 440.

[5] Barabási A.-L. and Albert R., *Science*, **286** (509) 1999.

[6] Barahona M. B. and Pecora L. M., *Phys. Rev. Lett.*, **89** (2002) 054101; Wang X. F. and Chen G., *Int. J. Bifurcat. Chaos Appl. Sci. Eng.*, **12** (2002) 187; Nishikawa T., Motter A. E., Lai Y.-C. and Hoppensteadt F. C., *Phys. Rev. Lett.*, **91** (2003) 014101.

[7] See, for example, Dezsõ Z. and Barabási A.-L., *Phys. Rev. E*, **65** (2002) 055103; Boguñá M., Pastor-Satorras R. and Vespignani A., *Phys. Rev. Lett.*, **90** (2003) 028701.

[8] Wang W.-X., Wang B.-H., Hu B., Yan G. and Ou Q., *Phys. Rev. Lett.*, **94** (2005) 188702; Wang W.-X. and Chen G., *Phys. Rev. E*, **77** (2008) 026101.

[9] Restrepo J. G., Ott E. and Hunt B. R., *Phys. Rev. E*, **71** (2005) 036151.

[10] See, for example, Restrepo J. G., Ott E. and Hunt B. R., *Phys. Rev. Lett.*, **100** (2008) 058701.

[11] May R., *Nature*, **238** (413) 1972; Feng J., Jirsa V. K. and Ding M., *Chaos*, **16** (2006) 015109.

[12] Cvetkovic D. and Rowlinson P., *Linear Multilinear Algebra*, **28** (1990) 3; MacCluer C. R., *SIAM Rev.*, **42** (2000) 487.

[13] Barrat A. *et al.*, *Proc. Natl. Acad. Sci. U.S.A.*, **101** (2004) 3747.

[14] Shen-Orr S., Milo R., Mangan S. and Alon U., *Nat. Genet.*, **31** (2002) 64.

[15] Jeong H., Mason S., Barabási A.-L. and Oltvai Z., *Nature*, **411** (2001) 41.

[16] Milo R. *et al.*, *Science*, **303** (2004) 1538.

[17] `http://www-personal.umich.edu/~mejn/netdata/`.

[18] Lusseau D., Schneider K., Boisseau O. J., Haase P., Slooten E. and Dawson S. M., *Behav. Ecol. Sociobiol.*, **54** (2003) 396.

[19] Girvan M. and Newman M. E. J., *Proc. Natl. Acad. Sci. U.S.A.*, **99** (2002) 7821.

[20] Zachary W. W., *J. Anthropol. Res.*, **33** (1977) 452.

[21] `http://www.orgnet.com/cases.html`.

[22] Newman M. E. J., *Proc. Natl. Acad. Sci. U.S.A.*, **98** (2001) 404.

[23] Newman M. E. J., *Phys. Rev. E*, **74** (2006) 036104.

[24] Cohen R., Erez K., ben-Avraham D. and Havlin S., *Phys. Rev. Lett.*, **85** (2000) 4626.

[25] Cvetković D. M., Doob M. and Sachs H., *Spectra of Graphs* (VEB Deutscher Verlag der Wissenschaften, Berlin) 1980.

[26] Kuramoto Y., *Chemical Oscillations Waves and Turbulence* (Springer-Verlag, Berlin) 1984.

[27] Bollobás B., Borgs C., Chayes J. and Riordan O., e-print arXiv.math/0701346.

[28] Arenas A., Díaz-Guilera A., Kurths J., Moreno Y. and Zhou C., *Phys. Rep.*, **469** (2008) 93.

[29] Zhou C., Motter A. E. and Kurths J., *Phys. Rev. Lett.*, **96** (2006) 034101.