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Effects of average degree on cooperation in networked evolutionary game

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Abstract. We study effects of average degree on cooperation in the networked prisoner's dilemma game. Typical structures are considered, including random networks, small-world networks and scale-free networks. Simulation results show that the average degree plays a universal role in cooperation occurring on all these networks, that is the density of cooperators peaks at some specific values of the average degree. Moreover, we investigated the average payoff of players through numerical simulations together with theoretical predictions and found that simulation results agree with the predictions. Our work may be helpful in understanding network effects on the evolutionary games.

PACS. 02.50.Le Decision theory and game theory – 89.75.Hc Networks and genealogical trees – 89.65.-s Social and economic systems – 05.10.-a Computational methods in statistical physics and nonlinear dynamics

1 Introduction

In natural and social systems, one of the most stunning phenomena is the ubiquitous cooperative behavior among selfish individuals, since defection actions usually bring much more benefits. Yet, understanding emergence and persistence of cooperation remains a challenge, which has drawn many interests from natural and social scientists [1–4]. So far, game theory has provided a powerful framework to characterize and investigate the evolution of cooperation [5,6]. One simple game, Prisoner's Dilemma game (PDG), has been considered as a general metaphor for studying cooperation among identical and unrelated individuals [7–10]. In the PDG, individuals can either cooperate or defect to play the game; If they mutually cooperate, both get reward R; while mutual defection results in punishment P to both. If one player cooperates while the other defects, the cooperator gains the lowest sucker's payoff S, while the defector gets the highest payoff, the temptation to defect T. Accordingly, the benefit order is $T > R > P > S$. However, mutual cooperation in the original one-shot PDG is unstable due to the highest payoff of defectors, which is in sharp contrast to real observations. Thus much effort has been paid to explain such contradiction.

Since the groundwork on repeated or iterated PDGs by Axelrod, much attention has been given to the repeated games and their suitable extensions, for cooperation can be obtained in repeated games under some specific conditions [4,10]. "Tit-for-tat" is a typical strategy which can remarkably enhance cooperative behavior in repeated games [4,11]. Another well-known rule leading to the high cooperation level is the "win stay and lose shift" [12]. Interestingly, an original work by Nowak and May [13] reported that the PDG with a simple spatial structure can induce emergence and persistence of cooperation, especially the observed spatial chaos. Enlightened by this idea, there has been a continuous effort on the effects of several types of structures on the cooperative behavior, such as regular graphs [14–21] and complex networks [22–30,21]. Understanding the effects of networks on the evolutionary games taking place on them has been judged to be one of the main goals in the study of games [9]. A surprising finding is that cooperation is inhibited in the snowdrift game (SG) by the spatial structure [31], which is sharp contrary to one's intuition, since the SG favors cooperation compared to the PDG. In a recent paper, Santos and Pacheco found that scale-free networks provide a unifying framework for the emergence of cooperation [32,33], which reveals that heterogeneous degree distribution plays a significant role in the cooperative behavior. However, how the other structural properties influence the cooperation and which one contributes to the emergence and persistence of cooperation remain unclear and need further study.

In the present work, we focus on the influence of the average degree of networks on the evolution of the

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networked PDG. By adopting three types of networks, i.e., scale-free Barabási-Albert networks (BA) [34], small-world Newman-Watts networks (NW) [35] (a variant of Watts-Strogatz small-world model [36]) and Erdös-Rényi random graphs (ER) [37], we figured out that there exist optimal values of the average degree for each kind of network leading to the best cooperation level. In parallel, we found that the average payoff of individuals is also a non-monotonous function of the average degree. Correspondent theoretical predictions are provided for the average payoffs of all individuals. The present study indicates that the average degree plays a universal role in evolutionary games on three types of networks, i.e., scale-free, small-world and random networks.

The paper is organized as follows. In the next section we describe the evolutionary game model as well as three types of networks in detail. In Section 3 simulation results and correspondent theoretical predictions are provided, and in Section 4 the work is summarized.

2 Model

Firstly, we construct networks using three typical models — the BA, NW and ER Models. The former two possess scale-free and small-world structural properties, respectively. Here, all the network sizes are set to $N = 5000$ in all simulations for convenient comparison. The crucial structural feature, average degree $\langle k\rangle$, can be adjusted by model parameters in these models. In the BA model, the parameter m, which denotes the number of edges of a new nodes attached to the existent networks at each time step, has a relation with degree, i.e., $\langle k\rangle = 2m$ [34]. In the NW network, a parameter p controls the fraction of edges randomly added to the regular ring graph. The relationship between p and $\langle k\rangle$ is $\langle k\rangle = 2(m_0 + p)$, where m_0 represents the coordination number of each node on the ring graph [35], we set $m_0 = 1$. In the ER network, $\langle k\rangle$ depends on a parameter p_{ER}, which characterizes the probability of establishing an edge between any pair of nodes, where the dependence of $\langle k\rangle$ on p_{ER} is $\langle k\rangle = Np_{ER}$ [37].

After constructing networks, each site of the network is occupied with an individual. An individual can be either a cooperator or a defector. All pairs of connected individuals play the game simultaneously and gain benefits according to the payoff parameters mentioned in the introduction. Here, following previous work, we adopt the rescaled version of payoffs as $R = 1$, $P = S = 0$, $T = b$ ($1 < b < 2$), such that the game is controlled by a single parameter b for convenient investigation. The total payoff of a certain player is calculated by summing the payoffs over all its interactions at each time step. During the evolutionary process, each player is allowed to learn from one of its neighbors and update its strategy at each round. The probability of a node i selecting one of its neighbors j is

$$\Pi_{i\to j} = \frac{k_j}{\sum_l k_l}, \tag{1}$$

where the sum runs over the set of neighbor nodes of i. The assumption of Π takes into account the fact that individuals with more interactions usually cause more attraction in society. In other words, well-known persons will have more influences than the others.

Whereafter, the node i will adopt the selected neighbor's strategy with a probability determined by the normalized payoff difference between them [18], i.e.,

$$W = \frac{1}{1 + \exp[(E_i/k_i - E_j/k_j)\beta]}, \tag{2}$$

where k_i and k_j respectively represent the degrees of node i and j. E_i and E_j respectively represent the total payoff of node i and j, and β characterizes the noise introduced to permit irrational choices. Here, β is set to 50. According to the evolutionism, W reflects the rule of natural selection based on relative fitness. Besides, the ratio of total income of a player and its degree E_i/k_i is defined as the normalized total payoff to avoid additional bias caused by the variety of degrees.

3 Results

One of the key quantities for characterizing the cooperative behavior is the density of cooperators ρ_c, which is defined as the fraction of cooperators in the whole population. We study ρ_c as a function of the average degree $\langle k\rangle$ for three types of networks. In all simulations, ρ_c is obtained by averaging over last 5 000 time steps of the entire 10 000 time steps and each data point results from 10 different network realizations. Initially, strategies C and D are uniformly distributed among all players. In Figure 1, we report ρ_c as a function of $\langle k\rangle$ for different values of b on BA, NW and ER networks, respectively. One can find that ρ_c exhibits a non-monotonous behavior with a peak at some specific values of $\langle k\rangle$. The larger value of b corresponds to the lower value of ρ_c at the peak point, which can be easily understood by noting the fact that large value b favors selfish action, leading to the reduction of the cooperation level. The common nontrivial behavior shared by the scale-free, small-world and random networks indicate that the average degree is a crucial feature for the networked evolutionary game.

Whereafter, we explain the non-monotonous dependence of ρ_c on $\langle k\rangle$. Actually, the effect of connectivity density on the cooperation over regular structures has been investigated previously [33,38]. Our work can be considered as expanding previous investigations to non-regular graphs. It has been known that the increase of the neighborhood of individuals drives the extinction of cooperators for a wide range of dynamics [38]. In our study, in the case of very large $\langle k\rangle$, the system reproduces the mean-field type behavior, so that the poor cooperation level is inevitable. On the other hand, the study of the PDG on one-dimensional networks has indicated that the cooperation is strongly inhabited with only nearest neighbor interactions [33]. Hence, for very low $\langle k\rangle$, cooperation will die out, since the system is close to a one-dimensional system. Combining the discussion of the two limits of $\langle k\rangle$, there should exist an optimal value of ρ_c in the middle range of

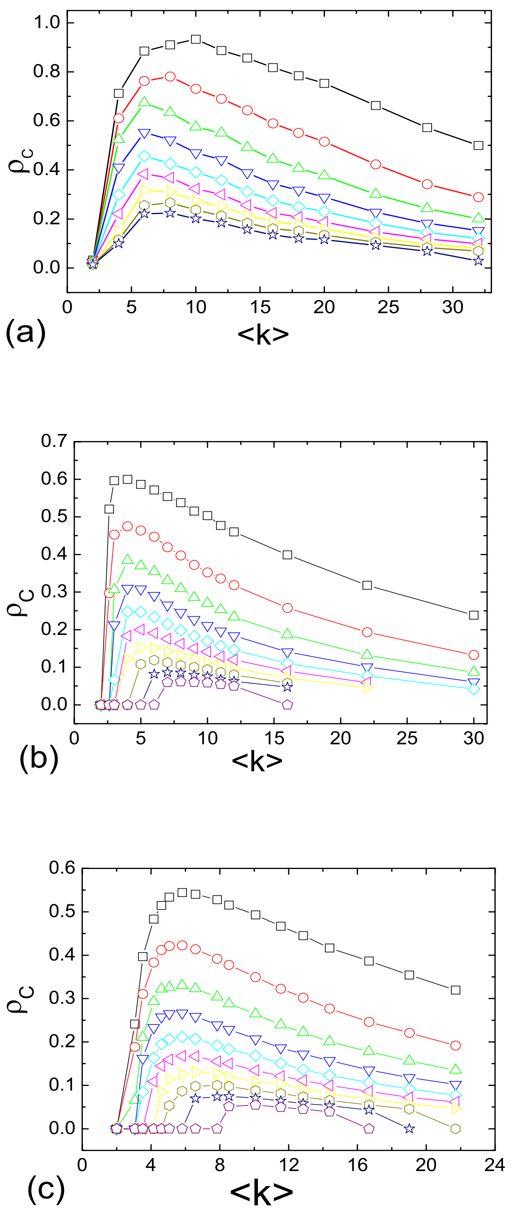

Fig. 1. (Color online) Cooperator density ρ_c vs average degree $\langle k\rangle$ on the (a) BA, (b) NW, and (c) ER networks for different values of the parameter b. Simulation were carried out for network size $N = 5000$. For the BA model, b ranges from 1.05 to 1.45 with a 0.05 interval; and for NW and ER networks, b ranges from 1.025 to 1.25 with a 0.025 interval. The upward curve corresponds to small value of b.

$\langle k\rangle$, for ρ_c is quite poor in the limits of both very low and high values of $\langle k\rangle$. Then, we study the maximum cooperation level ρ_c^{max} as a function of b for each type of network. As shown in Figure 2, ρ_c^{max} displays a decreasing trend for each network and shows approximately the same decreasing velocity for NW and ER networks, which reveals

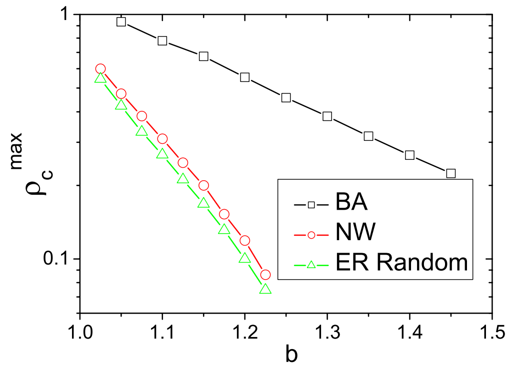

Fig. 2. (Color online) Log-normal plot of the peak value of density of cooperators ρ_C^{max} vs the parameter b in PDG for the three types of networks.

that these two kinds of networks may have very similar evolutionary dynamics.

We further concern the relationship between the average degree $\langle k\rangle$ and the average payoff $\langle M\rangle$ in the whole population for different types of networks (BA, NW, ER Networks). The average payoff is defined as

$$\langle M\rangle = \frac{1}{N}\sum_{i=1}^{N} E_i, \qquad (3)$$

where E_i is the total income of individual i. Also we found similar non-monotonous phenomena exhibited in the dependence of ρ_c on $\langle k\rangle$. Simulation results of $\langle M\rangle$ versus $\langle k\rangle$ for different values of b with adopting the BA, NW and ER network are reported in Figure 3. A phenomenon should be noted that the optimal value of $\langle k\rangle$ corresponding to the peak point of $\langle M\rangle$ is much higher than that corresponding to the best cooperation ρ_c^{max} in Figure 1. This phenomenon can be explained by noting a fact that more interactions usually bring more benefits even though altruistic action will result from the increment of $\langle k\rangle$. As $\langle k\rangle$ further increases from the maximum point in Figure 1, individuals will have more co-players, such that gain more benefits from the game with all their counterparts. Thus the augmentation of $\langle k\rangle$ will contribute to the average payoff of the whole population. On the other hand, as exhibited in Figure 1, increasing $\langle k\rangle$ induces more defection actions, which will reduce the income of defectors' cooperator neighbors. Since the positive effect of increasing $\langle k\rangle$ is much stronger than the caused benefit loss of individuals, the optimal value of $\langle k\rangle$ at which $\langle M\rangle$ peaks is larger than the value of $\langle k\rangle$ corresponding to ρ_c^{max}. Furthermore, it is also found that the maximum average payoff $\langle M\rangle$ of the BA scale-free network is nearly 4 times larger than the other two we've investigated. This result may due to the higher cooperation level in the scale-free network.

In the following, we provide theoretical predictions for the average payoff of individuals $\langle M\rangle$ by assuming cooperators are distributed uniformly among the network.

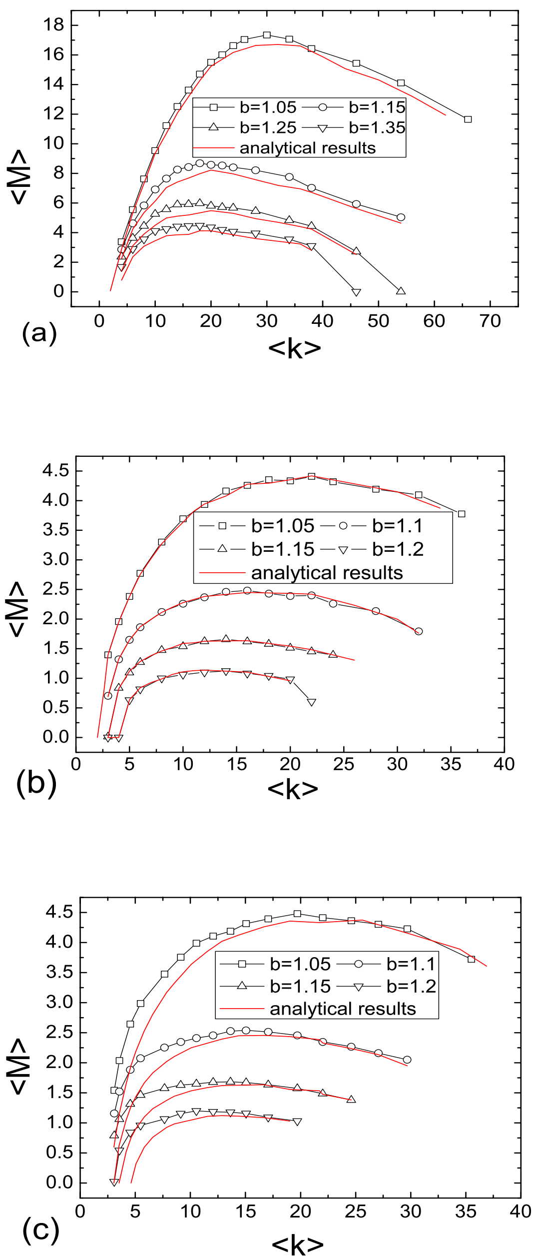

Fig. 3. (Color online) The average payoff $\langle M \rangle$ vs. the average degree $\langle k \rangle$ for different values of parameter b in the cases of the (a) BA, (b) NW, and (c) ER networks. Symbols are the simulation results and curves are the correspondent theoretical predictions. The network size N is 5000.

$\langle M \rangle$ can be expressed as

$$\langle M \rangle = (1 - \rho_c) \times \langle k \rangle \times \rho_c \times b + \rho_c \times \langle k \rangle \times \rho_c \times 1, \quad (4)$$

where the first term in the right side is the average payoff of defectors and the second term is the average payoff of cooperators. Equation (4) can be simplified to

$$\langle M \rangle = \langle k \rangle \times \rho_c \times ((1 - \rho_c) \times b + \rho_c). \quad (5)$$

Since the density of cooperators ρ_c cannot be reproduced by the mean-field approach, except for the well-mixed cases(fully connected networks), ρ_c used in equation (5) for calculating $\langle M \rangle$ is obtained by simulations, as shown in Figure 1. The comparison between simulation results and theoretical predictions is shown in Figure 3. In the case of NW networks, analytical results are in very good agreement with numerical ones. While in the cases of the BA, in particular the ER network, theoretical predictions are not in good accordance with simulations for low values of $\langle k \rangle$. The analytical results suggest that our approximation with neglecting pair correlations is suitable for small-world networks and for large average degrees of other types of networks.

4 Conclusion

We have studied the evolution of cooperation in the prisoner's dilemma game affected by the average degree of different types of networks. We found the average degree plays a universal role in the cooperation level in all types of investigated networks, i.e., the density of cooperators is a non-monotonous function of the average degree with the cooperator density peaks at some specific values of the average degree. We have given a qualitative explanation for this phenomenon. We have further studied the dependence of the average payoff of individuals on the average degree, and the similar non-monotonous behavior are observed. Correspondent theoretical predictions are provided. Analytical results are well consistent with numerical simulations in the case of small-world networks, while in the cases of scale-free and random networks, there are some difference between theoretical and numerical results for small average degrees. In terms of systematically investigations, we have clarified the effect of average connectivity on the cooperate behavior over three types of networks. Interestingly, the average connectivity plays a non-trivial role in the cooperation, i.e, there exists the optimum average connectivity resulting in the highest cooperation level. our work makes some contribution in the process of understanding network effects on the evolution of cooperation, which still deserves further efforts.

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