Synchronizability of Highly Clustered Scale-Free Networks *

WU Xiang(吴翔), WANG Bing-Hong(汪秉宏)**, ZHOU Tao(周涛), WANG Wen-Xu(王文旭), ZHAO Ming(赵明), YANG Hui-Jie(杨会杰)

Department of Modern Physics and Nonlinear Science Center, University of Science and Technology of China, Hefei 230026

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We consider the effect of clustering coefficient on the synchronizability of coupled oscillators located on scale-free networks. The analytic result for the value of clustering coefficient aiming at a highly clustered scale-free network model, the Holme–Kim model is obtained, and the relationship between network synchronizability and clustering coefficient is reported. The simulation results strongly suggest that the more clustered the network, the poorer the synchronizability.

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Many social, biological, and communication systems can be properly described as complex networks with nodes representing individuals and edges mimicking the interactions among them.[1−3] There have been examples such as the Internet, the World Wide Web, social networks, metabolic networks, food webs, and many others.[4−7] Recent empirical studies indicate that the networks in various fields have some common characteristics, the most important one of which are called the small-world effect[8] and scale-free property.[9] Networks of the small-world effect have small average distances as random networks and large clustering coefficient as regular ones. The scale-free property means that the degree distribution of networks obeys the power-law form.

One of the ultimate goals of research on complex networks is to understand how the structure of complex networks affects the dynamical process taking place on them, such as traffic flow,[10−13] epidemic spread,[14−16] cascading behaviour.[17−19] In this Letter, we concentrate on the synchronization, which is observed in a variety of natural, social, physical and biological systems.[20−22] The large networks of coupled dynamical systems that exhibit synchronized state are subjects of great interest. Previous studies mainly focus on the Watts–Strogatz[8] networks and Barabási–Albert[9] networks, which have demonstrated that scale-free and small-world networks are much easier to synchronize than regular lattices.[23−26] Since many real-life networks are scale-free small-world networks, there have already been some models that can simultaneously reproduce the small-world and scale-free characteristics,[27−30] to investigate the synchronizability of the scale-free small-world networks if of great interest and importance.

Since the scale-free networks are always of very small average distances,[31] the scale-fee small-world networks can also be referred as highly clustered scale-free networks. One of the earliest highly clustered scale-free models is the Holme–Kim (HK) model,[27] which has successfully reproduced the indirectly acquainting mechanism in real networks thus is closer to reality than the Barabási–Ansatz (BA) model. In this Letter, we present an analytical result about the clustering coefficient of the HK model, which is helpful for understanding the underlying evolution mechanism of the HK model. Then we investigate the relationship between the network synchronizability and clustering coefficient based on the HK model.

As a remark, previous studies mainly concentrate on how the average distance and heterogeneity of degree/betweenness distribution affect the network synchronizability,[24,25,32−36] while there are few systemic works about the effect of clustering coefficient. Although there are a number of highly scale-free models, the HK model is a typical one which has tunable clustering coefficient thus provides us a good researching stage. This is the reason why we choose the HK model as our theoretic template.

The HK network is generated by the following processes: (1) In each step, m edges are added in the networks, and t is a discrete parameter which denotes the global time that the system totally moves. (2) An edge is add with the probability $\Pi(k_i)$,

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}. \quad (1)$$

(3) Then, in the following $m-1$ time steps, do a PA (preferential attachment) step with the probability p or a TF (triad formation) step with the probability $1-p$ (see Ref. [27] for details).

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** To whom correspondence should be addressed. Email: bhwang@ustc.edu.cn

By using the rate-equation,[37] one can obtain the evolution of nodes' degree as follows:

$$\frac{\partial k_i(t)}{\partial t} = \Big\{1 - \Big(1 - \frac{k_i(t-1)}{\sum_{j=1}^{t-1} k_j(t-1)}\Big)^{m_{PA}}\Big\} + \Big\{1 - \Big(1 - \frac{\sum_{l\in\Omega_i} k_l(t-1)\frac{1}{k_l(t-1)}}{\sum_{j=1}^{t-1} k_j(t-1)}\Big)^{m_{TF}}\Big\}, \tag{2}$$

where Ω_i denotes the set of neighbours of node i, $k_i(t)$ is the degree of node i at time step t. In the above formula, m_{TF} is the number of edges that is connected by following the rule of triad formation in each step while m_{PA} denotes the number of those connected by following the rule of preferential attachment. Denote $k_i := k_i(N)$, where N is the network size, the two terms in the right side of the above formula are the degree increment rate of node i in PA and TF steps, respectively. By using the initial condition $k_i(t_i) = m$, and the expressions of $m_{PA} = (m-1)(1-p)+1$ and $m_{TF} = p(m-1)$, one can obtain the solution as follows:

$$k_i(t) = k_i(t; t_i, m), \tag{3}$$

from which and by using of the continuum theory,[36] one can obtain

$$P(k,p) = \frac{2m^2}{k^3} + A\frac{m^2}{k^2 N}, \tag{4}$$

where A is a quadric polynomial of p. Clearly, $p(k) = k^{-3}$ in the limit case $N \to \infty$.

The clustering coefficient of the whole network is the average of c_i over all nodes i, where c_i is the ratio between the number of edges among node i's neighbours which is denoted by n_i and the total possible number. Then,

$$c_i = \frac{2n_i}{k_i(k_i-1)}. \tag{5}$$

Using the rate-equation approach,[37] the detailed expression of $c(k)$, which denotes the average clustering coefficient over all the k-degree nodes, should be as follows:

$$c(k) = \frac{1}{k(k+1)}\frac{2m_{PA}m_{TF}}{m}(k-m) + \frac{1}{k(k+1)}\Big\{\frac{2m_{PA}m_{TF}}{m}(k-m) + \frac{m_{PA}(m_{PA}-1)}{16m}\frac{(\ln N)^2}{N}k^2\Big\}. \tag{6}$$

Here we assume that m_0 (the number of initial nodes) is equal to m (the edges added each time step). The three items in the right side in the above expression is obtained from the three mechanisms shown in Fig. 1.

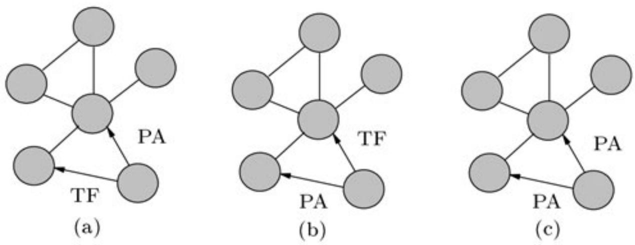

Fig. 1. The three parts represent three mechanisms: (1) node i is connected by a PA step while its neighbours are also connected by a TF step; (2) one of the neighbours of node i is connected by a PA step while the node i is connected by a TF step; (3) node i as well as one of its neighbours is connected by a PA step respectively.

Thus the clustering coefficient C can be solved as a function of the free parameter p,

$$c(p) = \int_{k_{\min}}^{k_{\max}} c(k)P(k)\mathrm{d}k. \tag{7}$$

In the above formula, $k_{\max} \to 2m\sqrt{N}$ and $k_{\min} = m$.[38] For $p \in [0,1]$, $c(p)$ can be simplified in a linear approximation.

$$c(p) = B(m,N) + c(m,N)p. \tag{8}$$

The extensive simulation results with different p and c for networks of different sizes strongly support the analytic results, especially in the larger-size networks, as shown in Fig. 2.

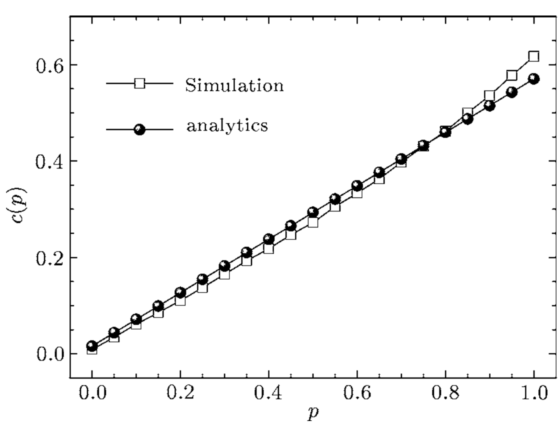

Fig. 2. Clustering coefficient $c(p)$ versus the parameter p in the HK model. The squares represent the simulation result while the circles denote the analytic solution. The network size is $N = 5000$, and $m = m_0 = 10$.

Other simulations have been carried out on the variation of average path length l and standard deviation of degree distribution σ. The results show that both the average path length l and standard deviation of degree distribution σ behave slightly variation with p. The intuitive explanation is quite easily understood that σ is directly related with degree distribution, that is to say, it is determined totally by the degree distribution, and the degree distribution will hardly vary when is N large enough. Thus we have the reason to neglect the effect of the two structural properties on synchronizability.

Here, we concern the system of linear coupled limit-cycle oscillators on HK networks. Describing the state of the ith oscillator by x_i, the equations of motion governing the dynamics of the N coupled oscillators are

$$\dot{x}_i = F(x_i) + K\sum_{j=1}^{N} M_{ij}G(x_j), \tag{9}$$

where $\dot{x}_i = F(x_i)$ characterizes the dynamics of individual oscillators, $G(x_j)$ is the output function, K denotes the coupling strength and the $N\times N$ coupling matrix M is

$$M_{ij} = \begin{cases} -k_i, & \text{for } i=j \\ 1, & \text{for } j\in\Lambda_i \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

In the above expression, Λ_i denotes the neighbours of node i. Because of the negative semidefinition and the zero sum of each raw of the matrix, all its eigenvalues are nonpositive real values and the largest eigenvalue λ_0 is always zero. Thus the eigenvalues can be ranked as $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_N$, and $\lambda_0 = \lambda_1 = 0$ if and only if the network is disconnected.

In our coupled dynamic network, all the oscillators are identical and the same output function is used, the coupling fashion ensures that the synchronization manifold is an invariant manifold and the nodes can be well approximated near the synchronous state by a linear operator. Under these conditions, the eigenratio $R = \frac{\lambda_N}{\lambda_2}$ can be used to measure the network synchronizability; the smaller the R value, the stronger the synchronizability.[25,33–35,39–46]

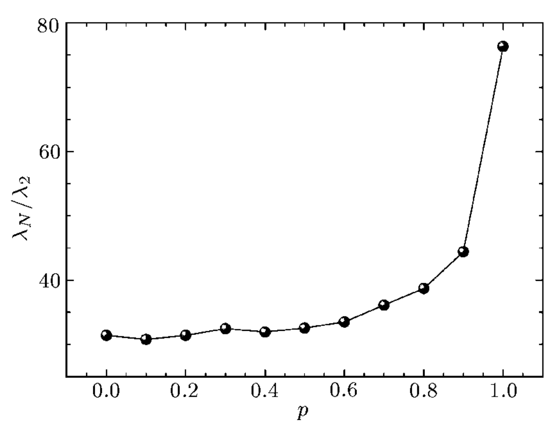

Fig. 3. Synchronizability R measured by $\frac{\lambda_N}{\lambda_2}$ versus the parameter p in the HK model. The simulation has been carried out under the condition that the size of the networks is $n = 1800$. Other parameters are $m = m_0 = 10$.

We take only synchronizability $R(p)$ and clustering coefficient $c(p)$ into consideration. Having simulated $R(p)$ for different configurations versus p (see Fig. 3), we know that $R(p)$ is positively correlated with p, and so is $c(p)$ although it is not completely linear with p. In addition, we can easily find that when p is not very large, the curve is approximately linear (see Fig. 4).

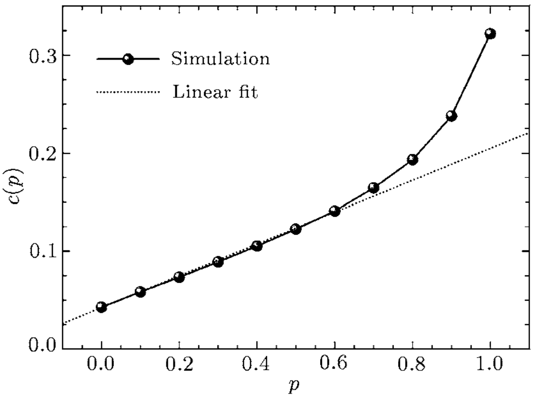

Fig. 4. Clustering coefficient $c(p)$ versus parameter p under the condition of $N = 1800$, and $m = m_0 = 10$. The linear fitted line is for the first seven points.

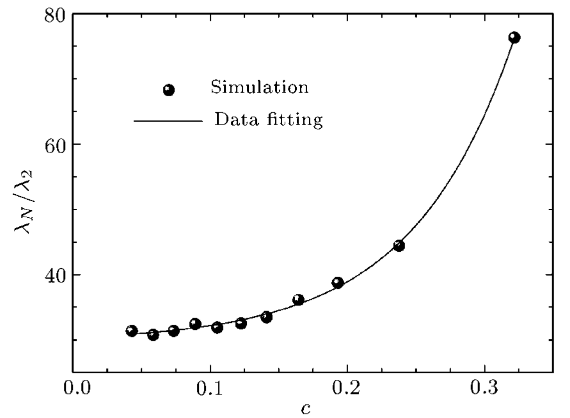

Fig. 5. Synchronizability R measured by $\frac{\lambda_N}{\lambda_2}$ as a function of clustering coefficient c in the case of $N = 1800$, and $m = m_0 = 10$. The fitted curve follows the tendency of exponential growth.

Furthermore, we report the relationship between synchronizability $R(p)$ and clustering coefficient $c(p)$ as shown in Fig. 5. From the figure, one important fact the curve reveals is exponential growing tendency. The function of fitted curve can be set as follows:

$$R(c) = A(m,N) + B(m,N)e^{\frac{c}{T(m,N)}}. \tag{11}$$

The terms $A(m,N), B(m,N)$, and $T(m,N)$ can be obtained by simulation. Here the simulation averaged over 50 different realizations is to measure the effect of random fluctuation of degree distribution, since the degree distribution can both affect l and σ. In our simulation, due to the average effect, the two parameters, l and σ do not vary significantly so that we can only focus on clustering coefficient exclusively which causes the change of synchronizability.

Compared to the previous works, the advantages of the HK model is that the clustering coefficient can be tuned while the other structural properties are almost kept to be fixed. Therefore, combining the behaviours of $c(p)$ versus p and $R(p)$ versus p, we obtain the relationship between synchronizability and clustering coefficient. Figure 4 demonstrates that the larger the clustering coefficient, the poorer the synchronizability. Due to the fact that the synchronizability R is determined by the ratio of maximal and minimal eigenvalues of coupling matrix which is exclusively related to the network topology, moreover c is the only varied topological property, the clustering coefficient plays a crucial role in synchronizability. The size effect has not been discussed so far in this study. As the size N increases, the average distance becomes larger. As a result, the synchronization become harder, but any qualitative changes will presumably not occur, especially on the negative correlation between clustering and synchronizability. Preliminary simulation results strengthen this conjecture.

To ascertain the effect of each structural characteristics on synchronizability is a meaningful work because if future study can ascertain the relationships between each typical structural characteristics (l and σ etc.) and synchronizability exclusively, we can finally obtain the expression of synchronizability as functions of those properties. We still wonder whether a network can achieve synchronization and how synchronizability of a specific structure can be easily predicted only by the topological characteristics.

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