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2009 Chinese Phys. B 18 3122

(http://iopscience.iop.org/1674-1056/18/8/004)

Vol 18 No 8, August 2009
1674-1056/2009/18(08)/3122-09
Chinese Physics B

Are networks with more edges easier to synchronize, or not?*

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(Received 15 September 2008; revised manuscript received 25 November 2008)

In this paper, the relationship between network synchronizability and the edge-addition of its associated graph is investigated. First, it is shown that adding one edge to a cycle definitely decreases the network synchronizability. Then, since sometimes the synchronizability can be enhanced by changing the network structure, the question of whether the networks with more edges are easier to synchronize is addressed. Based on a subgraph and complementary graph method, it is shown by examples that the answer is negative even if the network structure is arbitrarily optimized. This reveals that generally there are redundant edges in a network, which not only make no contributions to synchronization but actually may reduce the synchronizability. Moreover, a simple example shows that the node betweenness centrality is not always a good indicator for the network synchronizability. Finally, some more examples are presented to illustrate how the network synchronizability varies following the addition of edges, where all the examples show that the network synchronizability globally increases but locally fluctuates as the number of added edges increases.

Keywords: complex network, complementary graph, synchronizability, edge addition
PACC: 0250, 0565, 0545

1. Introduction and problem formulation

Systems composed of dynamical units are ubiquitous in nature, ranging from physical to technological, and to biological fields. These systems can be naturally described by networks with nodes representing the dynamical units and links representing the interactions among them. The topologies of such networks have been extensively studied and some common architectures have been discovered.[1−3] The small-world property, for example, characterized by short average distance and high clustering among nodes, is one of the most common properties shared by many real networks.[4] More significantly, many networks show high heterogeneity of node connectivity, which typically possesses a power-law distribution, named scale-free networks.[5] It is known that these topological characteristics have a strong influence on the dynamics of the structured systems, such as epidemic spreading, traffic congestion, collective synchronization, and so on.[6,7] From this viewpoint, systematically understanding the network structural effects on their dynamical processes is of both theoretical and practical importance.

Synchronization behavior, in particular, as a widely observed phenomenon in networked systems, has received a great deal of attention in the past few decades.[8−15] Oscillator network models have been commonly used to characterize synchronization behaviors. Based on the master stability function method,[16−18] one knows that two factors influence the network synchronizability: one is related to the eigenvalues of the Laplace matrix of the network, and the other is related to the synchronized region, which is determined by the node dynamics and the inner-linking function of the network. The product of the network coupling strength and the eigenvalues of the Laplace matrix plays an important role in characterizing the network synchronizability: the more easily it falls into the synchronized region, the more

*Project supported by the National Natural Science Foundation of China (Grant Nos 10832006 and 60674093), the Foundation for Key Program of Educational Ministry, China (Grant No 107110) and the City University of Hong Kong under the Research Enhancement Scheme and SRG (Grant No 9041335).
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easily the network achieves synchronization, namely, the larger the network synchronized region, the easier the synchronization of the network. On the other hand, for a network with given node dynamics and a fixed inner-linking function, the synchronized region has been determined. It is found that, in this case, when the synchronized region is unbounded, the smallest nonzero eigenvalue of the Laplace matrix determines the network synchronizability; when the synchronized region is bounded, the ratio of the smallest nonzero eigenvalue to the largest eigenvalue of the Laplace matrix determines the network synchronizability. In this framework, it has been found that a smaller average distance does not necessarily imply better synchronizability.[19] And the node betweenness centrality was provided as a good indicator to the network synchronizability.[20] Since the synchronizability is correlated with many topological properties, a natural question is which property is the most significant to the synchronizability? Donetti *et al*[21] tried to answer this question by an optimization argument. They believed that a network with optimized synchronizability should have an extremely homogeneous structure, i.e., the distributions of some fundamental topological properties should be very narrow. However, a recent result[22] shows that the network synchronizability has no direct relationship with the statistical properties. Therefore, this is still a difficult question to answer, although it has been studied by many authors.[19–24] Perhaps a good index which can characterize the synchronizability has not been found. In recent years, graph-theoretic methods were used to analyze the network synchronizability, e.g., degree sequences were discussed;[25] complementary graph and subgraph techniques were used;[22,26] it was shown that the expectation of the largest eigenvalue can be well approximated by the lower bound $d_{\max}+1$ in random scale-free networks,[27] and different bounds for the eigenvalues of the Laplace matrix and the synchronizability index were established.[28,29]

Motivated by the above work, this paper focuses on the relationships between the network synchronizability and the edge-addition of the associated graph. The effects of the connection patterns of graphs on the synchronizability are analyzed both theoretically and numerically. It is found that adding an edge to a cycle of size $N \geq 5$ definitely decreases the network synchronizability, but the synchronizability may be improved by changing the cyclic structure. However, a further example shows that, by arbitrarily optimizing the network structures, networks with more edges are not necessarily easier to synchronize. This implies that there are redundant edges in the network with respect to synchronization.

Consider a dynamical network consisting of N coupled identical nodes, with each node being an n-dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}H(x_j),\ i=1,2,\ldots,N, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ is the state vector of node i, $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth vector-valued function, constant $c > 0$ represents the coupling strength, $H(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called the inner linking function, and $A = (a_{ij})_{N\times N}$ is called the outer coupling matrix or topological matrix, which represents the coupling configuration of the entire network. This paper only considers the case that the network is diffusively connected, i.e., the entries of A satisfy

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij},\ i=1,2,\ldots,N.$$

Further, suppose that if there is an edge between node i and node j, then $a_{ij} = a_{ji} = -1$, i.e., A is a Laplace matrix. If the graph corresponding to A is connected, then 0 is an eigenvalue of A with multiplicity 1, and all the other eigenvalues of A are strictly positive, which are denoted by

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_N. \tag{2}$$

The dynamical network (1) is said to achieve (asymptotical) synchronization if $x_1(t) \rightarrow x_2(t) \rightarrow \ldots \rightarrow x_N(t) \rightarrow s(t)$, as $t \rightarrow \infty$, where, because of the diffusive coupling configuration, the synchronous state $s(t) \in \mathbb{R}^n$ is a solution of an individual node, i.e., $\dot{s}(t) = f(s(t))$.

For given node dynamics and inner-linking function, if the synchronized region is bounded, then the eigenratio $r(A) = \lambda_2/\lambda_N$ of the network structural matrix A characterizes the synchronizability. The larger the $r(A)$ is, the better the synchronizability will be. The enhancement of the network synchronizability and the relationships between $r(A)$ and the network structural characteristics, such as average distance, node betweenness, degree distribution, clustering coefficient, etc., have been well studied.[11,22,23,25] This paper further investigates the relationship between the network edges and its synchronizability by graph-theoretical tools.

Throughout this paper, for any given undirected graph G, eigenvalues of G mean eigenvalues of its corresponding Laplace matrix. Notations for graphs and their corresponding Laplace matrices are not differentiated, and networks and their corresponding graphs are not distinguished, unless otherwise indicated.

2. Adding one edge to a cycle

It has been known that more edges do not necessarily imply a better synchronizability,[22] and it was found[24] that in scale-free networks where the nodes are coupled symmetrically, if some overloaded edges are removed, the network will become more synchronizable. This section gives a definite result for one edge addition to cycles. To show this, the following lemmas are needed.[25,30,31]

Lemma 1 For any given connected graph G of size N, its nonzero eigenvalues indexed as listed in formula (2) grow monotonically with the number of added edges; that is, for any added edge e, $\lambda_i(G+e) \geq \lambda_i(G)$, $i = 1, \ldots, N$.

Lemma 1 shows the eigenvalue changes of graphs due to the addition of edges, but it does not show any information about the eigenratio $r(A)$. Therefore, this eigenratio needs to be studied in more detail.

Lemma 2 For any given connected graph G of size N, its largest eigenvalue λ_N satisfies $\lambda_N \geq d_{\max} + 1$, with equality if and only if $d_{\max} = N - 1$. Further, if G is not a complete graph, then the smallest nonzero eigenvalue of G satisfies $\lambda_2 \leq d_{\min}$. Here $d_{\max}$ and $d_{\min}$ denote the maximum and minimum degrees of G.

Lemma 3 For any cycle C_N with N (≥ 4) nodes, its eigenvalues are given by $\mu_1, \ldots, \mu_N$ (not necessarily ordered as in formula (2)) with $\mu_1 = 0$ and

$$\mu_{k+1} = 3 - \frac{\sin\left(\dfrac{3k\pi}{N}\right)}{\sin\left(\dfrac{k\pi}{N}\right)},\ k = 1, \ldots, N-1.$$

Lemma 4 Given a connected graph G, if the multiplicity of its smallest nonzero eigenvalue λ_2 is larger than or equal to 2, then adding one edge to G cannot change this eigenvalue, i.e., $\lambda_2(G+e) = \lambda_2(G)$.

Proof This lemma follows from the fact that $\mathrm{rank}(\lambda_2 I - (G+e)) \leq \mathrm{rank}(\lambda_2 I - G) + 1$. □

By the above lemmas, one can obtain the following result for cycles.

Theorem 1 For any cycle C_N with $N \geq 4$ nodes, adding one edge cannot enhance its synchronizability $r(C_N)$; specifically, one has $r(C_4 + e) = r(C_4)$ and $r(C_N + e) < r(C_N)$ ($N \geq 5$).

Proof $r(C_4+e) = r(C_4)$ holds obviously. For the case of $N \geq 5$, by Lemma 2, one has $\lambda_N(C_N + e) > 4$. But by Lemma 3, $\lambda_N(C_N) \leq 4$. And Lemma 3 shows that the multiplicity of the smallest nonzero eigenvalue λ_2 of C_N is 2. By Lemma 4, $\lambda_2(C_N + e) = \lambda_2(C_N)$. Therefore, $r(C_N + e) < r(C_N)$ for all $N \geq 5$. □

Theorem 1 shows that adding one edge to a cycle with $N \geq 5$ nodes definitely decreases the network synchronizability, as shown by the two examples in Figs.1–5. By simple computation, one obtains $r(C_5) = \frac{1.3820}{3.6180} = 0.3820$ and $r(C_5 + e\{1,3\}) = \frac{1.3820}{4.6180} = 0.2993 < r(C_5)$; $r(C_6) = \frac{1}{4} = 0.25$, $r(C_6 + e\{1,3\}) = \frac{1}{4.4142} = 0.2265 < r(C_6)$ and $r(C_6 + e\{1,4\}) = \frac{1}{5} = 0.2 < r(C_6)$.

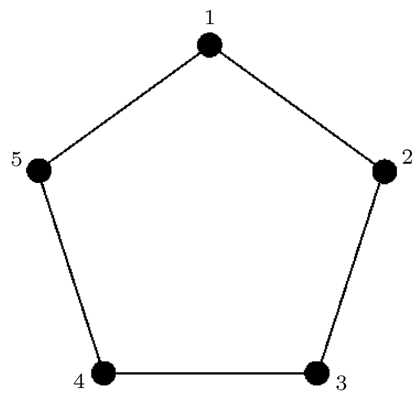

Fig.1. Graph G_5.

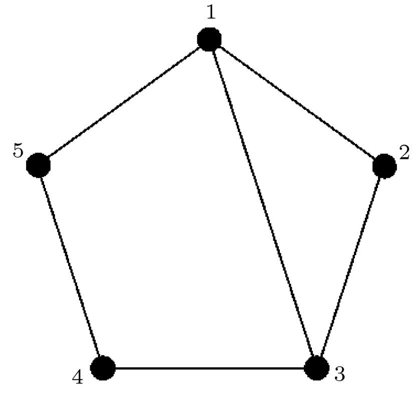

Fig.2. Graph $C_5 + e\{1,3\}$.

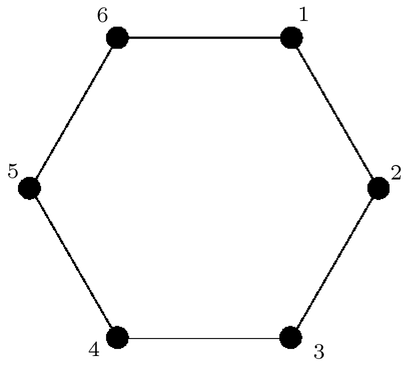

Fig.3. Graph C_6.

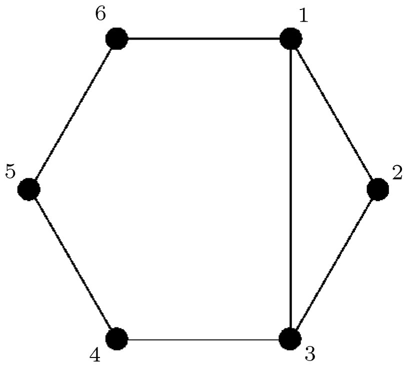

Fig.4. Graph $C_6 + e\{1,3\}$.

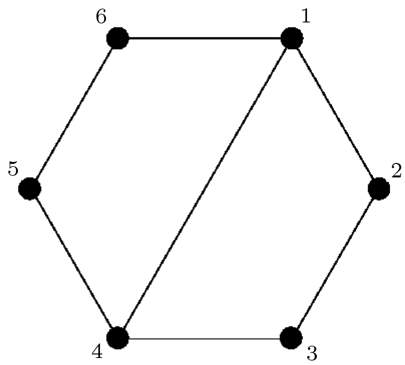

Fig.5. Graph $C_6 + e\{1,4\}$.

From the above two examples, one can find that the synchronizability of cycles strictly decreases if only one edge is added, and the results vary depending upon where the edge is placed, e.g., $r(C_6 + e\{1,3\}) > r(C_6 + e\{1,4\})$. Considering the optimization of network structures, $r(C_6 + e\{1,3\})$ is still not the best one among all the graphs with 7 edges connecting 6 nodes in a cycle, as demonstrated in the next section.

3. Changing the network structure to enhance its synchronizability

It is shown in the above section that adding one edge to a cycle decreases its synchronizability. A further question is whether the synchronizability can be enhanced by changing the network structure after edge addition. The answer is 'yes' in some cases. For example, one can change $C_5+e\{1,3\}$ to C_{5o} as in Fig.6, and $C_6+e\{1,3\}$ to C_{6o} as in Fig.7. Then, $r(C_{5o}) = \frac{2}{5} = 0.4$ and $r(C_{6o}) = \frac{1.2679}{4.7231} = 0.2684$.

Comparing the graphs in Figs.1–5, one can see that both the synchronizabilities of C_{5o} and C_{6o} have been improved. In fact, two cycles share a common edge in Figs.2, 4 and 5. In this case, generally the betweenness centrality is large, or the node-to-node distances are not homogeneous. In comparison, the network structural characteristics are more homogeneous in Figs.6 and 7. This is consistent with the result of Ref.[21]. For simple graphs with a few nodes and edges, as those shown above, one can compute their eigenvalues to find a good structure for the synchronizability. However, for a general graph, how does one optimize the network structure toward the best possible synchronizability? Some optimal rules are provided based on an optimizing algorithm:[21] to have homogeneous degree, node distance, betweenness, and loop distributions. But these rules are observed from simulations, and theoretical proofs are not available by now. And, sometimes, these rules are contrary to each other. For example, comparing $C_6 + e\{1,3\}$ with $C_6 + e\{1,4\}$, one can find that the cycle of $C_6+e\{1,4\}$ is more homogeneous, but the average node distance of $C_6+e\{1,3\}$ is smaller. It seems that the importance of these rules should be ordered.

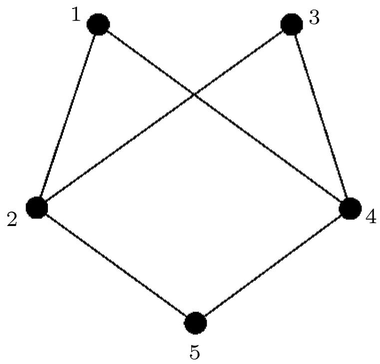

Fig.6. Graph C_{5o}.

Although some rules are provided,[21] optimizing the network structure for better synchronizability is still a hard problem, since it is possible that the optimizing algorithm converges to a suboptimal solution.

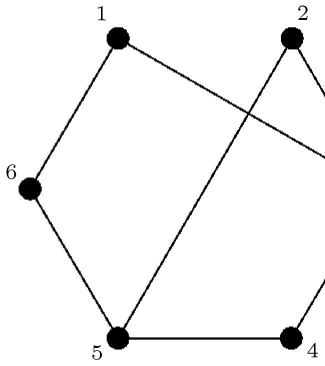

Fig.7. Graph C_{6o}.

Other than the rules for optimization, complementary graphs can be used to characterize the network synchronizability.[22] For a given graph G, the complementary graph of G, denoted by G^c, is the graph containing all the nodes of G and all the edges that are not in G. For eigenvalues of graphs and complementary graphs, the following lemma is useful (see Refs.[30], [31] and references therein).

Lemma 5 For any given graph G, the following statements hold:

(i) $\lambda_N(G)$, the largest eigenvalue of G, satisfies $\lambda_N(G) \leq N$.

(ii) $\lambda_N(G) = N$ if and only if G^c is disconnected.

(iii) If G^c is disconnected and has (exactly) q connected components, then the multiplicity of $\lambda_N(G) = N$ is $q-1$.

(iv) $\lambda_i(G^c) + \lambda_{N-i+2}(G) = N$, $2 \leq i \leq N$.

For example, the complementary graph of C_{5o} is disconnected (see Fig.8) and the largest eigenvalue of C_{5o} is 5, the number of nodes. Its smallest nonzero eigenvalue $\lambda_2 = 2$ can be easily obtained by computing the largest eigenvalue of its complementary graph. Further, according to the complementary graph, adding one more edge to graph C_{5o} cannot enhance its synchronizability. However, if adding two more edges to C_{5o}, e.g., $e\{1,5\}$ and $e\{3,5\}$, then the synchronizability increases to $r = \frac{3}{5}$. The corresponding complementary graph becomes the complementary graph of cycle C_4 (see Fig.9). Cycle C_4 and its complementary graph are very important in graph theory[31] (see the section below for their further applications).

For a given graph G, if its complementary graph is disconnected and includes two separated graphs G_1 and G_2, then by Lemma 5 the synchronizability of G is $r(G) = (N - \max\{\lambda_{\max}(G_1), \lambda_{\max}(G_2)\})/N$, where N is the number of nodes of G and $\lambda_{\max}$ denotes the maximum eigenvalue of the corresponding Laplace matrix. It is well known that the complementary graphs of bipartite graphs are disconnected,[22,32] so the synchronizability of bipartite graphs can be simply analyzed by the above method. Actually, C_{5o} in Fig.6 is a bipartite graph. Obviously, better understanding and careful manipulation of complementary graphs are useful for enhancing the network synchronizability (see the section below for further applications of complementary graphs).

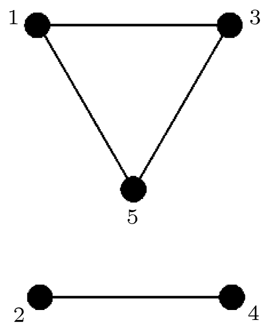

Fig.8. Graph C_{5o}^c.

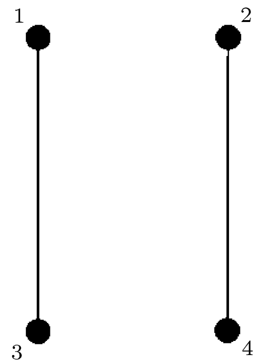

Fig.9. Graph C_4^c.

4. Are networks with more edges easier to synchronize?

For a given graph G, let $\mathcal{V}$ and $\mathcal{E}$ denote the sets of nodes and edges of G, respectively. A graph G_1 is called an induced subgraph of G, if the node set $\mathcal{V}_1$ of G_1 is a subset of $\mathcal{V}$ and all the edges of G_1 are the edges in $\mathcal{E}$. In this section, subgraphs and complementary graphs are used to discuss network synchronizability.

Concerning the optimization of network structures, an interesting question is whether networks with more edges are easier to synchronize. In order to answer this question, the following lemma is needed.

Lemma 6 For any given graph G, suppose G_1 is its induced subgraph including all nodes of G with the maximum degree $d_{\max}$. If G_1 includes a cycle C_{2k} with even nodes $2k$ ($k \geq 2$) is an induced subgraph, then the largest eigenvalue of G satisfies $\lambda_N(G) \geq d_{\max}+2$.

Proof By Lemma 3, for any cycle C_{2k} with even nodes, its largest eigenvalue is 4. And since the degree of every node is 2, -2 must be an eigenvalue of the adjacency matrix of C_{2k}. Let L_1 be the sub-matrix of the Laplace matrix of G associated with all the nodes in G_1. By this assumption, one has

$$(d_{\max}+2)I - L_1 = 2I + A(G_1) \not> 0,$$

where $A(G_1)$ is the adjacency matrix of G_1. This implies that the largest eigenvalue of G_1 is larger than or equal to $d_{\max}+2$. Thus, Lemma 1 leads to the result directly. □

Remark 1 Besides Lemma 2, there are few results on the lower bounds of the largest eigenvalue of Laplace matrices in graph theory.[30,32] Since networks with good synchronizability always have homogeneous degree distributions, Lemma 6 is very useful for the study of network synchronization.

Theorem 2 For any connected graph G with 16 edges on 10 nodes, its eigenratio is bounded by $r(G) < 2/5$.

Proof If the largest node degree of G is $d_{\max} \geq 6$, then the smallest node degree must satisfy $d_{\min} \leq 2$. The conclusion follows directly from Lemma 2. In order to have good synchronizability, the degree distribution of G should be homogeneous. Then, first suppose that G has 8 nodes with degree 3 and two nodes with degree 4. In this case, by Lemma 2 the largest eigenvalue of G is $\lambda_{10}(G) > 5$ (G is connected).

In what follows, consider the largest eigenvalue of the complementary graph G^c. By the above discussion, G^c must have 8 nodes with degree 6 and two nodes with degree 5. Suppose G_1 is the subgraph of G^c composed of 8 degree-6 nodes. By direct computing, G_1 must have 19 or 20 edges, and the degree of every node is at least 4. Hence, G_1^c has 9 or 8 edges and the degree of every node is at most 3. If the largest eigenvalue of G_1 is 8, i.e., G_1^c is disconnected (Lemma 5), then the largest eigenvalue of G^c is larger than or equal to 8. Therefore, the smallest nonzero eigenvalue of G is $\lambda_2(G) \leq 10-8=2$. By the above discussion, the theorem obviously holds. According to the edges and node degrees, G_1 must be connected. Hence, suppose both G_1 and G_1^c are connected. Then, G_1 must have a cycle C_4 as an induced subgraph. This holds if and only if G_1^c has C_4^c (see Fig.9) as an induced subgraph. With only 9 or 8 edges having a node degree at most 3, drawing G_1^c directly one can easily reach the conclusion. By Lemma 6, the largest eigenvalue of G^c must be larger than or equal to 8. Repeating the above discussion concludes the proof.

When G has 9 nodes with degree 3 and one node with degree 5, the proof can be similarly completed. □

Remark 2 Theorem 2 shows that there is not a graph G with 16 edges on 10 nodes whose synchronizability is $r(G) \geq 2/5$. However, there does exist a graph Γ_1 with 15 edges on 10 nodes whose synchronizability is $r(\Gamma_1) = 2/5$ (see Fig.10), consistent with the result of Ref.[21]. This clearly shows that networks with more edges are not necessarily easier to synchronize. In fact, by the optimal result of Ref.[21], $r = 2/5$ is the optimal synchronizability for graphs with 15 edges on 10 nodes. For any graph G with 16 edges on 10 nodes, if both G and G^c have cycles with even nodes, then by Lemma 6 and Theorem 2, $r(G) \leq 2/6 = 1/3$. Therefore, adding one more edge definitely decreases the synchronizability. The existence of cycles with even nodes can be easily tested by drawing graphs, so Lemma 6 is very useful for analyzing the synchronizability of homogeneous networks. Actually, the graph shown in Fig.10 is quite homogeneous in structure.[21] With one more edge being added, such a structure is destroyed. It is therefore easy to understand why adding more edges does not necessarily result in better synchronizability.

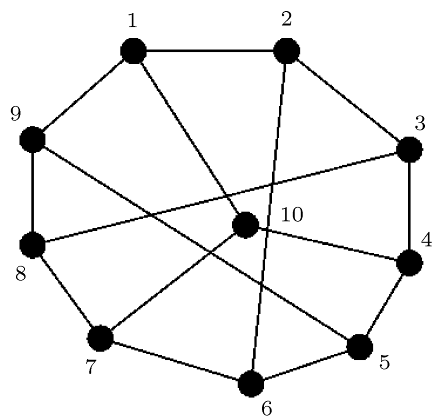

Fig.10. Graph Γ_1, $r(\Gamma_1) = 2/5$.

Remark 3 Figure 11 shows a new graph Γ_2 with 20 edges on 10 nodes. It also has quite homogeneous structural characteristics as discussed in Ref.[21]. In fact, the betweenness centrality of each node of Γ_1 is 6, larger than that of Γ_2, 5. But, the synchronizability of graph Γ_2 is worse than that of graph Γ_1, contrary to the result of Ref.[21]. So far, the existing theories[19−23] cannot explain why the synchronizability of Γ_1 is better than that of Γ_2. This shows the complexity of the relationships between the synchronizability and network structural characteristics. Although Γ_2 has the property of homogeneity, another question is whether there exists another graph with 20 edges on 10 nodes having better synchronizability than that of Γ_1 or Γ_2? If the answer is negative, it implies that generally there are many redundant edges in a network with respect to its synchronizability. These kinds of questions are still open today.

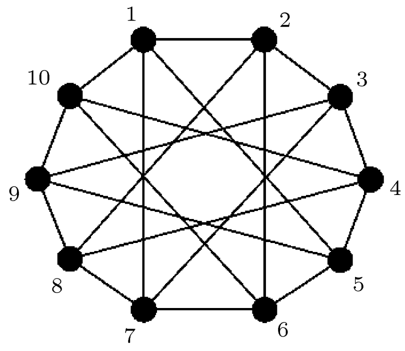

Fig.11. Graph Γ_2, $r(\Gamma_2) = \dfrac{2.7639}{7.2361} \approx 0.382 < r(\Gamma_1) = 2/5$.

5. Some examples

In this section, some examples are given to show the changes of the synchronizability versus the addition of edges.

Example 1 The synchronizability changes by adding edges to graphs with cycles are shown in Figs.12 and 13, where their initial graphs are C_{10} and C_{50}, respectively, and $m_{\rm add}$ denotes the number of added edges.

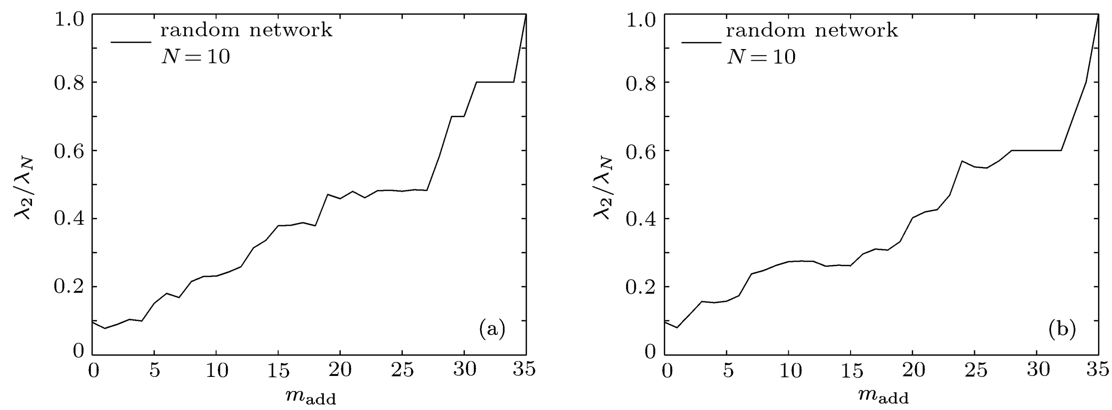

Fig.12. The synchronizability changes of graphs obtained from C_{10} by adding edges. (a) Adding edges with degree homogeneity, (b) randomly adding edges.

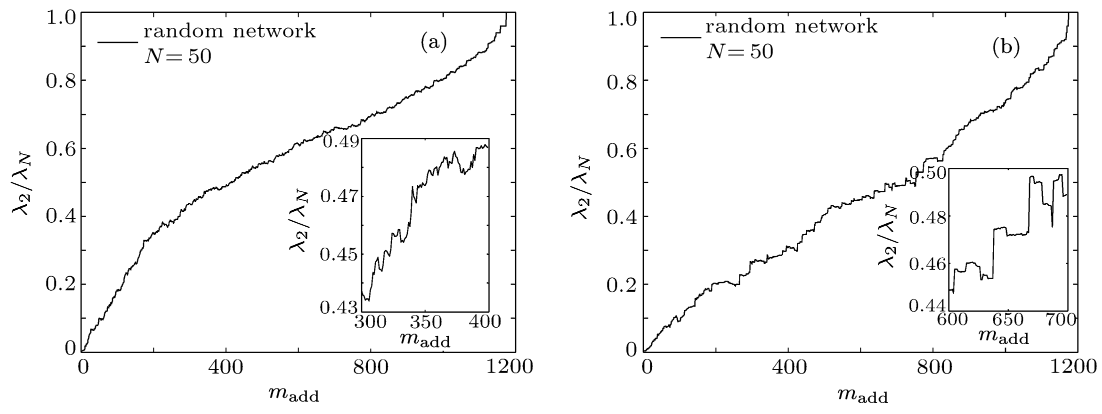

Fig.13. The synchronizability changes of graphs obtained from C_{50} by adding edges. (a) Adding edges with degree homogeneity, (b) randomly adding edges.

The figures in (a)s show the synchronizability changes during the process of adding edges with degree homogeneity (i.e., guaranteeing the node degrees be as homogeneous as possible during edge-adding). The figures in (b)s show the cases corresponding to random edge-adding. Naturally, the corresponding synchronizabilities in (a)s are better than those in (b)s, since degree homogeneity is an important property for networks to achieve good synchronizabilities. In all graphs, it is shown that the synchronizability globally increases but locally fluctuates. According to Theorem 2 and Remark 2, this is the expected phenomenon.

Example 2 The synchronizability changes of graphs obtained from a scale-free graph by randomly adding edges are shown in Fig.14, for which, the same conclusion can be drawn as in Example 1.

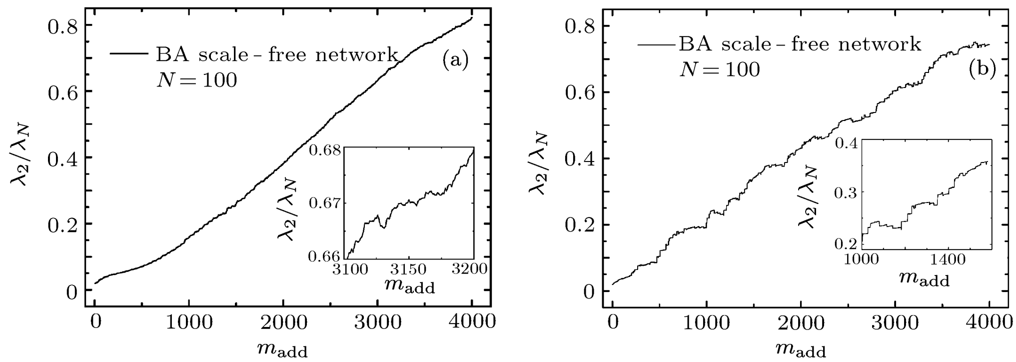

Fig.14. The synchronizability changes of graphs obtained from a scale-free graph by adding edges. (a) Adding edges with degree homogeneity, (b) randomly adding edges.

6. Conclusion

In recent years, complex networks have received great attention from various research areas.[1,3,8,33−37] In fact, complex networks are closely related to their corresponding graphs, e.g., the network synchronizability is closely related to the eigenvalues of the network Laplace matrix. The properties of Laplace matrix have been well studied in algebraic graph theory.[30−32] Obviously algebraic graph theory can be used to analyze network synchronization behaviors.[26,29] In this paper, the relationship between the network synchronizability and the edge distribution of the associated graphs has been further studied by graph tools. It has been proved that the synchronizability definitely decreases if one edge is added to a cycle with N ($N \geq 5$) nodes. However, it has also been shown that the synchronizability can be improved by changing the network structure. Further, examples have shown that some networks with more edges, unexpectedly, have worse synchronizabilities even if the network structures are in some sense optimized. This implies that, for network synchronization, generally there are redundant edges, which do not make any contribution to synchronization but may actually destroy the synchronizability. In addition, an example of a graph with 20 edges on 10 nodes has been provided to show that existing theories cannot explain why it has worse synchronizability than that of a graph with 15 edges on 10 nodes. Some other examples have also been given to show that the network synchronizability globally increases but locally fluctuates due to edge-adding. According to these results, in practical synchronization problems, the synchronizability and the number of communication edges should have a coordinative relation. And one may utilize the redundant edges to improve robustness or other network properties. These kinds of important questions remain open for further research in the future.

References

[1] Albert R and Barabási A L 2002 *Rev. Mod. Phys.* **74** 47

[2] Dorogovtsev S N and Mendes J F F 2002 *Adv. Phys.* **51** 1079

[3] Fang J Q, Li Y and Bi Q 2007 *Int. J. Mod. Phys.* B **21** 5121

[4] Watts D J and Strogatz S H 1998 *Nature* **393** 440

[5] Barabási A L, Albert R and Jeong H 1999 *Science* **286** 509

[6] Newman M E J 2003 *SIAM Rev.* **45** 167

[7] Boccaletti S, Latora V, Moreno Y, Chavez M and Hwang D U 2006 *Phys. Rep.* **424** 175

[8] Zhou J, Lu J A and Lü J H 2006 *IEEE Trans. Auto. Contr.* **51** 652
[9] Lu W L and Chen T P 2004 *Physica D* **198** 148
[10] Belykh I V, Lange E and Hasler M 2005 *Phys. Rev. Lett.* **94** 188101
[11] Liu Q, Fang J Q and Li Y 2007 *Commun. Theor. Phys.* **47** 752
[12] Zhou C, Motter A E and Kurths J 2006 *Phys. Rev. Lett.* **96** 034101
[13] Sorrentino F, di Bernardo M, Garofalo F and Chen G 2007 *Phys. Rev.* E **75** 046103
[14] Wang X F and Chen G 2002 *IEEE Trans. Circuits Syst.*- I **49** 54
[15] Liu Z R, Dong C D and Fan Q D 2007 *Int. J. Bifur. Chaos* **17** 2109
[16] Pecora L M and Carroll T L 1998 *Phys. Rev. Lett.* **80** 2109
[17] Stefański A, Perlikowski P and Kapitaniak T 2007 *Phys. Rev.* E **75** 016210
[18] Lü J H, Yu X H, Chen G and Cheng D Z 2004 *IEEE. Trans. Circuits Syst.*- I **51** 787
[19] Nishikawa T, Motter A E, Lai Y C and Hoppensteadt F C 2003 *Phys. Rev. Lett.* **91** 014101
[20] Hong H, Kim B J, Choi M Y and Park H 2004 *Phys. Rev.* E **69** 067105
[21] Donetti L, Hurtado P I and Muñoz M A 2005 *Phys. Rev. Lett.* **95** 188701
[22] Duan Z S, Chen G and Huang L 2007 *Phys. Rev.* E **76** 056103
[23] Zhao M, Zhou T, Wang B H, Yan G, Yang H J and Bai W J 2006 *Physica* A **371** 773
[24] Yin C Y, Wang W X, Chen G and Wang B H *Phys. Rev.* E **74** 047102
[25] Wu C W 2005 *Phys. Lett.* A **346** 281
[26] Duan Z S, Liu C and Chen G 2008 *Physica* D 237 1006
[27] Kim D H and Motter A E 2007 *Phys. Rev. Lett.* **98** 248701
[28] Motter A E 2007 *New. J. Phys.* **9** 182
[29] Comellas F and Gago S 2007 *J. Phys.* A: *Math. Theor.* **40** 4483
[30] Merris R 1994 *Linear Algebra Appl.* **197** 143
[31] Merris R 1998 *Linear Algebra Appl.* **278** 221
[32] Pan Y L 2002 *Linear Algebra Appl.* **355** 287
[33] Li Y, Cao H D, Shan X M and Ren Y 2008 *Chin. Phys.* B **17** 2327
[34] Pei W D, Chen Z Q and Yuan Z Z 2008 *Chin. Phys.* B **17** 373
[35] Zhao J, Tao L, Yu H, Luo J H, Cao Z W and Li Y X 2007 *Chin. Phys.* **16** 3571
[36] Guo J L 2007 *Chin. Phys.* **16** 1239
[37] Chen F, Che Z Q and Yuan Z Z 2007 *Chin. Phys.* **16** 287