Outline and Formulas

- The Language: Density matrix for both classical objects and quantum objects. For classical system: the equivalence between density matrix and probability distribution.

- Brief Review of Classical Game Theory, which is defined as

$$\Gamma^c = \left\langle N, \mathbb{S}, G \triangleq \left\{G^i\right\}\right\rangle \tag{1}$$

Payoff of player i is determined by

$$E^i\left(s^1, \cdots, s^N\right) = G^i\left(s^1, \cdots, s^N\right) \tag{2}$$

Nash Equilibrium, the goal of Game Theory, also called the solution of a game, is defined as $\rho_{eq}^1 \cdots \rho_{eq}^N$ as

$$E^i\left(\rho_{eq}^1 \cdots \rho_{eq}^i \cdots \rho_{eq}^N\right) \geq E^i\left(\rho_{eq}^1 \cdots \rho^i \cdots \rho_{eq}^N\right), \forall i, \forall \rho^i. \tag{3}$$

- Hilbert space of object states and Hilbert space of Operator, the inner product

$$\langle A|\, B\rangle \triangleq (A, B) = \frac{Tr\left(A^\dagger B\right)}{Tr\left(I\right)},$$

- Manipulative definition and abstract definition.

$$\Gamma^m = \left(\rho_0^o \in \mathcal{H}^o, S \triangleq \left\{s^i \in S^i\right\}, \mathcal{L}, P \triangleq \left\{P^i\right\}\right), E^i\left(\rho\right) = Tr\left(P^i \mathcal{L}\left(S\right) \rho_0^o \mathcal{L}^\dagger\left(S\right)\right). \tag{4}$$

$$\Gamma^a = \left(\rho^S \in \mathbb{S} \triangleq \prod_{i=1}^N \otimes S^i, H \triangleq \left\{H^i\right\}\right), E^i = Tr\left(\rho H^i\right). \tag{5}$$

General NE

$$E^i\left(\rho_{eq}^S\right) \geq E^i\left(Tr^i\left(\rho_{eq}^S\right)\rho^i\right), \forall i, \forall \rho^i \tag{6}$$

- Another way from Classical Game to Quantum Game, the probability distribution over the whole pure strategy space.

$$\rho^{S,q} = \sum_{S \in \mathbb{S}} p^{S,q}\left(S\right)\left|S\right\rangle\left\langle S\right|, \tag{7}$$

where $S = \left|s^1, s^2, \cdots, s^N\right\rangle$.

- Examples

First example, the Prisoner's Dilemma.

$$H^{1,2}=\left[\begin{array}{cccc} -4,-4 & 0 & 0 & 0 \\ 0 & 0,-5 & 0 & 0 \\ 0 & 0 & -5,0 & 0 \\ 0 & 0 & 0 & -2,-2 \end{array}\right], G^{1,2}=\left[\begin{array}{cc} -4,-4 & -5,0 \\ 0,-5 & -2,-2 \end{array}\right]$$

Second example, we choose Hawk-Dove, a two-pure-NE game,

$$H^{1,2}=\left[\begin{array}{cccc} 3,3 & 0 & 0 & 0 \\ 0 & 1,4 & 0 & 0 \\ 0 & 0 & 4,1 & 0 \\ 0 & 0 & 0 & 0,0 \end{array}\right], G^{1,2}=\left[\begin{array}{cc} 3,3 & 4,1 \\ 1,4 & 0,0 \end{array}\right]$$

Third example, Penny Flipping Game,

$$H^{1,2}=\left[\begin{array}{cccc} 1,-1 & 0 & 0 & 0 \\ 0 & -1,1 & 0 & 0 \\ 0 & 0 & -1,1 & 0 \\ 0 & 0 & 0 & 1,-1 \end{array}\right], G^{1,2}=\left[\begin{array}{cc} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{array}\right]$$

Manipulative Definition of Penny Flipping Game

$$\rho_0^o=\left|+1\right\rangle\left\langle+1\right|=\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \quad S^1=S^2=\{I,X\}, \quad \mathcal{L}=s^2s^1, \quad P^1=\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]=-P^2$$

Manipulative Definition of Spin Rotating Game

$$\rho_0^o=\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \quad S^1=S^2=\mathbb{E}\{I,X,Y,Z\}, \quad \mathcal{L}=s^2s^1, \quad P^1=\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]=-P^2$$

The abstract definition of Spin Rotating Game,

$$H^1=\left[\begin{array}{cccccccccccccccc}
1 & 0 & 0 & 1 & 0 & 1 & i & 0 & 0 & -i & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & -1 & -i & 0 & -1 & 0 & 0 & -1 & -i & 0 & 0 & -i & 0 & 1 & i & 0 \\
0 & i & -1 & 0 & i & 0 & 0 & i & -1 & 0 & 0 & -1 & 0 & -i & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & i & 0 & 0 & -i & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & -1 & -i & 0 & -1 & 0 & 0 & -1 & -i & 0 & 0 & -i & 0 & 1 & i & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & i & 0 & 0 & -i & 1 & 0 & 1 & 0 & 0 & 1 \\
-i & 0 & 0 & -i & 0 & -i & 1 & 0 & 0 & -1 & -i & 0 & -i & 0 & 0 & -i \\
0 & -1 & -i & 0 & -1 & 0 & 0 & -1 & -i & 0 & 0 & -i & 0 & 1 & i & 0 \\
0 & i & -1 & 0 & i & 0 & 0 & i & -1 & 0 & 0 & -1 & 0 & -i & 1 & 0 \\
i & 0 & 0 & i & 0 & i & -1 & 0 & 0 & 1 & i & 0 & i & 0 & 0 & i \\
1 & 0 & 0 & 1 & 0 & 1 & i & 0 & 0 & -i & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & i & -1 & 0 & i & 0 & 0 & i & -1 & 0 & 0 & -1 & 0 & -i & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & i & 0 & 0 & -i & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & i & 0 & 1 & 0 & 0 & 1 & i & 0 & 0 & i & 0 & -1 & -i & 0 \\
0 & -i & 1 & 0 & -i & 0 & 0 & -i & 1 & 0 & 0 & 1 & 0 & i & -1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & i & 0 & 0 & -i & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right].$$