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Waves induced by heterogeneity in oscillatory media

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Chunli Huang¹, Xiaoqing Huang², Xiaoming Zhang³ and Xiaohua Cui¹¹ School of Systems Science, Beijing Normal University, Beijing, 100875, People's Republic of China² School of Biomedical Engineering, Capital Medical University, Beijing, 100069, People's Republic of China³ College of Physics and Optoelectronic Engineering, Shenzhen University, Shenzhen 518060, People's Republic of ChinaE-mail: xhcui@bnu.edu.cn**Keywords:** nonlinear wave, pattern, complex Ginzburg–Landau equation, heterogeneitySupplementary material for this article is available [online](#)

Abstract

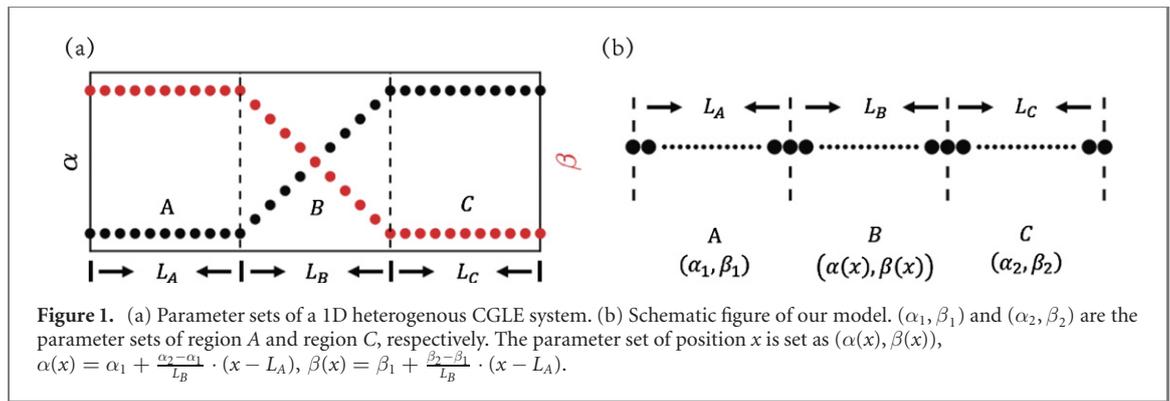
Various behaviours of nonlinear wave propagation and competition have been discussed and investigated extensively and meticulously, especially when the media are homogeneous. However, corresponding studies in heterogeneous media are much scarcer. In this paper, spontaneously generated waves from one-dimensional heterogeneous oscillatory media, modelled by complex Ginzburg–Landau equations with spatially varied controlling parameters, are investigated. An unexpected homogeneous wave train clearly emerges under certain conditions. With the theory of interface-selected waves, we can theoretically predict the frequencies and wavenumbers under several conditions. This kind of wave train can be found in a wide region of parameter space. These phenomena are robust when parameters are varied nonlinearly or linearly with fluctuation. Moreover, this kind of homogeneous wave plays an important role in wave competition and affects wave propagation in spatially heterogeneous nonlinear systems, which will bring new applications of heterogeneity and provide new ideas for wave control.

1. Introduction

A wide range of wave patterns can be sustained in spatially extended active systems, including physical, chemical [1, 2] and biological systems [3–6]. We are familiar with the behaviour of waves in most homogeneous systems, such as the propagation or competition of waves. Moreover, some novel behaviour around interfaces has been well predicted analytically and observed clearly, such as wave reflection and refraction in linear optics, negative refraction and interface-selected waves in nonlinear oscillatory systems [7–9]. The behaviours of waves has been explored and determined step by step. However, when the media become heterogeneous, the behaviour of waves becomes more complex and diverse, and little is known.

Spatial heterogeneity is one of the important factors that can strongly affect the selection of wave patterns and has attracted great interest during recent decades [10–19]. For example, it has been found that in excitable and oscillatory systems with low heterogeneity, coherent wave patterns were generated [11, 12]. Heterogeneity also can select new wave patterns [7, 15, 16]. Currently, the wave behaviour around the interface in inhomogeneous systems composed of two homogeneous systems has attracted considerable attention in nonlinear systems. Both planar wave and target wave patterns can be selected by the interface [12]. However, the knowledge of wave patterns selected in heterogeneous systems is still insufficient, so it is worth further investigation. For example, what kinds of pattern formations can be deduced from the heterogeneity, and will the kinds of heterogeneity (partial heterogeneity, spatial heterogeneity with parameters changing regularly or randomly, and so on) affect the pattern formation?

Intuitively, interface-selected waves (ISW) can be generated spontaneously from the interface of an inhomogeneous system constructed by one normal wave medium and one anti-wave medium, and the ISW is the result of interplay between two different media at the interface [7]. If one system is overall heterogenous in space, what kind of dynamical behaviour will emerge? In this paper, we study the selection of wave patterns in one-dimensional (1D) heterogeneous systems modelled by the complex



Ginzburg–Landau equation (CGLE). Spatially varied control parameters are selected to form a spatial heterogeneity. Surprisingly, a homogeneous pattern has been observed under different settings of spatial heterogeneity. The understanding and explanation of such interesting phenomena for heterogeneous systems are expected to extend to more complicated practical heterogeneous oscillatory systems and help us understand heterogeneity, which is of great significance in practice.

In the next section, a heterogeneous one-dimensional CGLE model is described in detail with dispersion relation curves for different media referring to the setting of the parameters. The variations of the parameters in space could be linear or nonlinear. The resultant wave patterns generated spontaneously in these systems are shown in section 3. The unexpected phenomena draw us to analyse the physical explanation of wave generation in section 4 and the robustness of ISWs in section 5. In the last section, we present an overall discussion and conclusion about the generation rules of wave patterns in these types of heterogeneous nonlinear systems.

2. Model

We consider a 1D oscillatory system modelled by the CGLE:

$$\frac{\partial A(x, t)}{\partial t} = A(x, t) - (1 + i\alpha(x))|A(x, t)|^2 A(x, t) + (1 + i\beta(x))\nabla^2 A(x, t) \quad (1)$$

with the complex variable $A(x, t)$ being the order parameter at a Hopf bifurcation [20–22]. For a general reaction–diffusion system, the dynamics of oscillations with the amplitude and phase are scaled to one complex order parameter $A(x, t)$. $\alpha(x)$ and $\beta(x)$ are real parameters, representing the nonlinear frequency shift and the dissipative coefficient, respectively. $\nabla^2 = \partial^2/\partial x^2$ denotes the 1D Laplace operator.

An inherent oscillation with natural frequency $\omega_0 = \alpha$ can automatically survive in a homogeneous CGLE system that is well known, and the inherent oscillations will compete with each other if the system is composed of several homogeneous CGLE systems around the interface and come to a final winner [21, 22].

A homogeneous CGLE medium can have plane wave solutions, and the frequency and wave number of wave trains satisfy the dispersion relation [21, 23]:

$$\omega = \omega_0 + f_1 k^2 = \alpha + (\beta - \alpha)k^2 \quad (2)$$

where $f_1 = \beta - \alpha$ is the slope of the dispersion relation curve in the $\omega - k^2$ plane, and $\omega_0 = \alpha$ is the actual frequency of the homogeneous no-flux system. The characteristic of propagating waves is determined by the dispersion relation curve and the frequency of the wave train (ω) [23]:

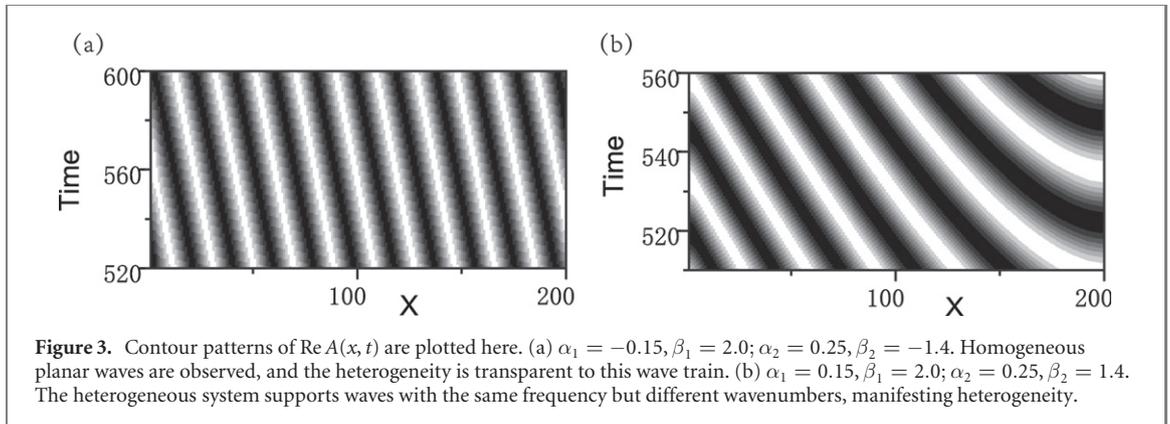
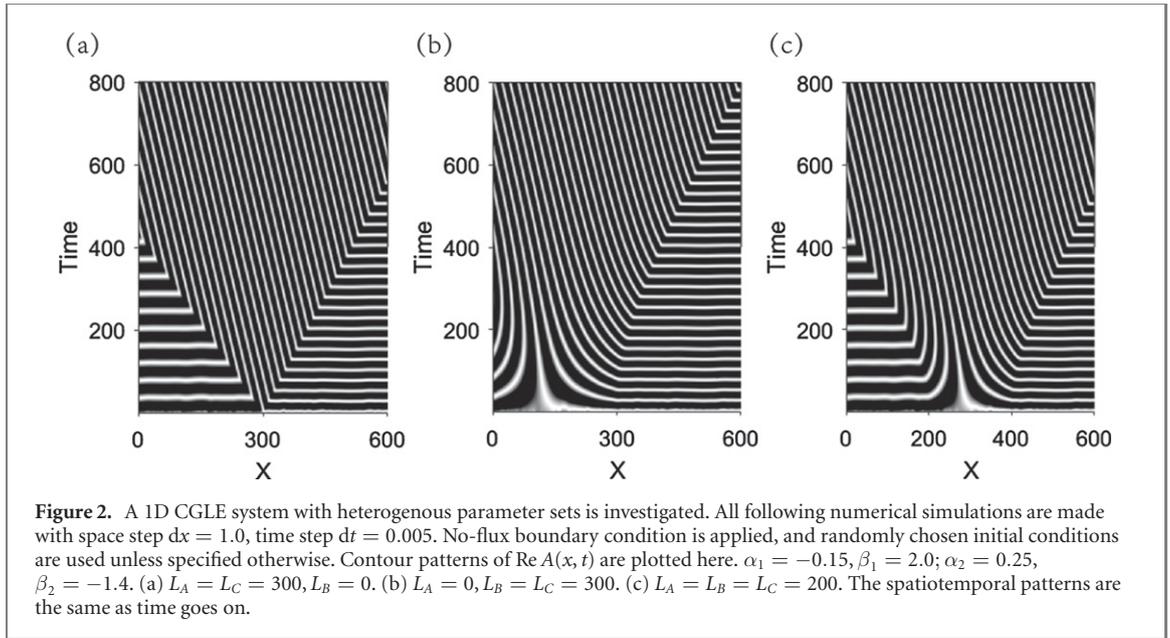
$$\text{NW} : \omega f_1 = \omega(\beta - \alpha) > 0 \quad (3a)$$

$$\text{AW} : \omega f_1 = \omega(\beta - \alpha) < 0 \quad (3b)$$

Here, we study a 1D CGLE system and consider a spatial heterogeneous system, as shown in figure 1. The parameters in region A are (α_1, β_1) , in region C are (α_2, β_2) . The parameters $(\alpha(x), \beta(x))$ in region B are varied via position x .

$$\alpha(x) = \alpha_1 + (\alpha_2 - \alpha_1) * (x - L_A)^n / L_B^n \quad (4a)$$

$$\beta(x) = \beta_1 + (\beta_2 - \beta_1) * (x - L_A)^n / L_B^n \quad (4b)$$



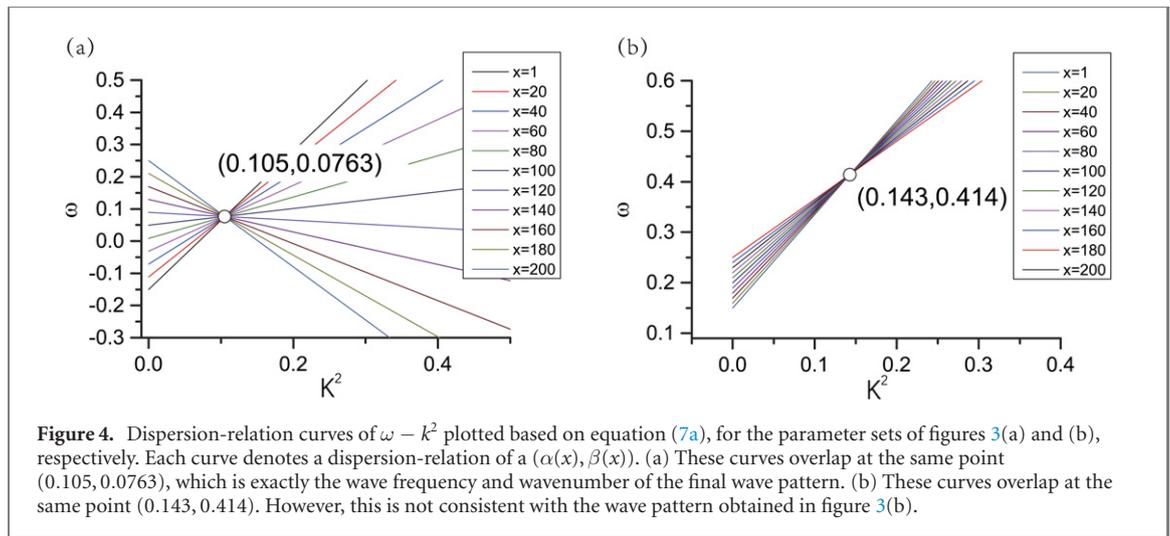
Since the parameters $\alpha(x)$ and $\beta(x)$ change via position x , we have different inherent oscillations at different spatial positions. By changing the parameter set of (α, β) properly, it might become possible to observe the detailed behaviour occurring exactly on these inherent oscillations and reveal the dynamic behaviour induced by the spatial heterogeneity.

The CGLE system here is integrated using the explicit Euler method and standard three-point approximation for the Laplace operator. Besides, no-flux boundary condition is applied, and randomly chosen initial conditions are used. Throughout the paper we simulate equation (1) with space step $dx = 1.0$, and time step $dt = 0.005$. We have confirmed the validity of the numerical method by using the Runge–Kutta method and smaller space step (e.g., $dx = 0.5$), and got results similar to those presented in the following figures without visible errors.

3. Pattern formation phenomena observed in heterogeneous CGLE systems

In order to know how heterogeneity significantly affects the inherent dynamics when no external pacing is involved. We plot figure 2 to illustrate the transition between two-homogeneous coupled media and a pure heterogeneous media. Regular wave patterns will be observed as time goes on. The results are almost the same, so here we focus on the pure heterogeneous system (i.e., $L_A = L_C = 0$).

We first study the parameter sets $(\alpha(x)$ and $\beta(x))$ by changing linearly via position x ; i.e., $n = 1$ in equation (4). The beginning of the heterogeneous system is set as (α_1, β_1) , and then α increases linearly while β decreases. At the end of the system, the parameters are (α_2, β_2) . In figure 3, we used the same parameter sets as above and set random initial conditions to study the system evolution. We found different features under asymptotic states. We show the most significant observation in figure 3(a). Homogeneous planar waves with a constant velocity are observed along with a planar wave generated by pacing in a



homogeneous medium. The phenomenon shown in figure 3(a) is surprising. With a totally spatial heterogeneous medium, we intuitively think that the waves may have the same frequency but must have different wave numbers. In figure 3(b), where we observe synchronistic oscillations in the medium with different wave numbers, the spatial heterogeneity is clearly manifested. In addition, in figure 3(a), waves propagate in the spatially heterogeneous media as if they are propagating in a homogeneous medium. Moreover, the generated running waves are stable for different initial perturbations. This phenomenon is impossible in linear systems and is still unusual in nonlinear systems.

4. Explanation of regular wave patterns

It is interesting that we can accurately predict the frequency and wavenumber of this kind of homogeneous wave under certain parameter conditions. For the case of figure 3(a), we can determine that the frequencies and wavenumbers at different positions have the same values:

$$\omega(x) = \omega_0; \quad k(x) = k_0; \quad |F(x)| = A_0; \quad 1 \leq x \leq L_B \quad (5)$$

and a single-domain planar wave solution [7, 20, 21] can be written as follows:

$$A(x, t) = F(x) \exp(i(k(x)x - w(x)t)), \quad 1 \leq x \leq L_B. \quad (6)$$

Inserting equations (5) into (6), we obtain a unique set of solutions $\omega(x)$, $k(x)$.

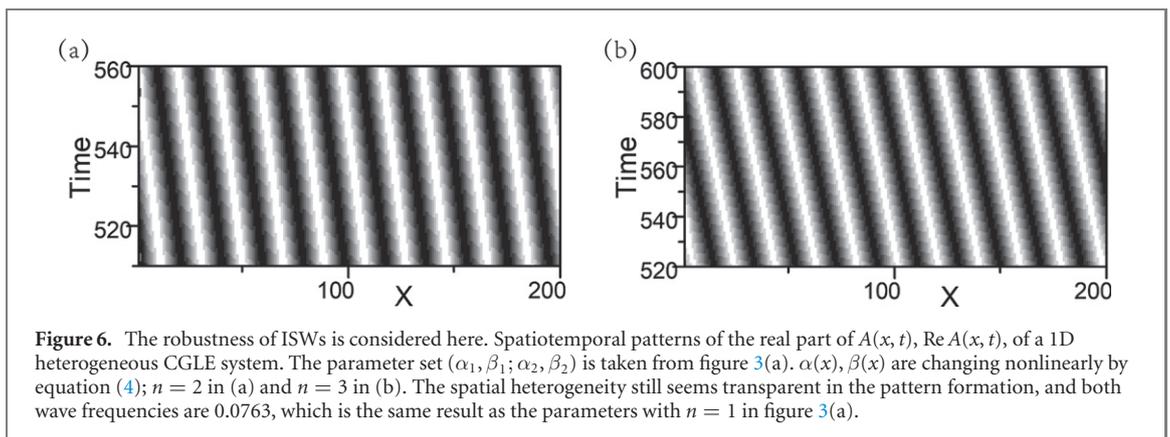
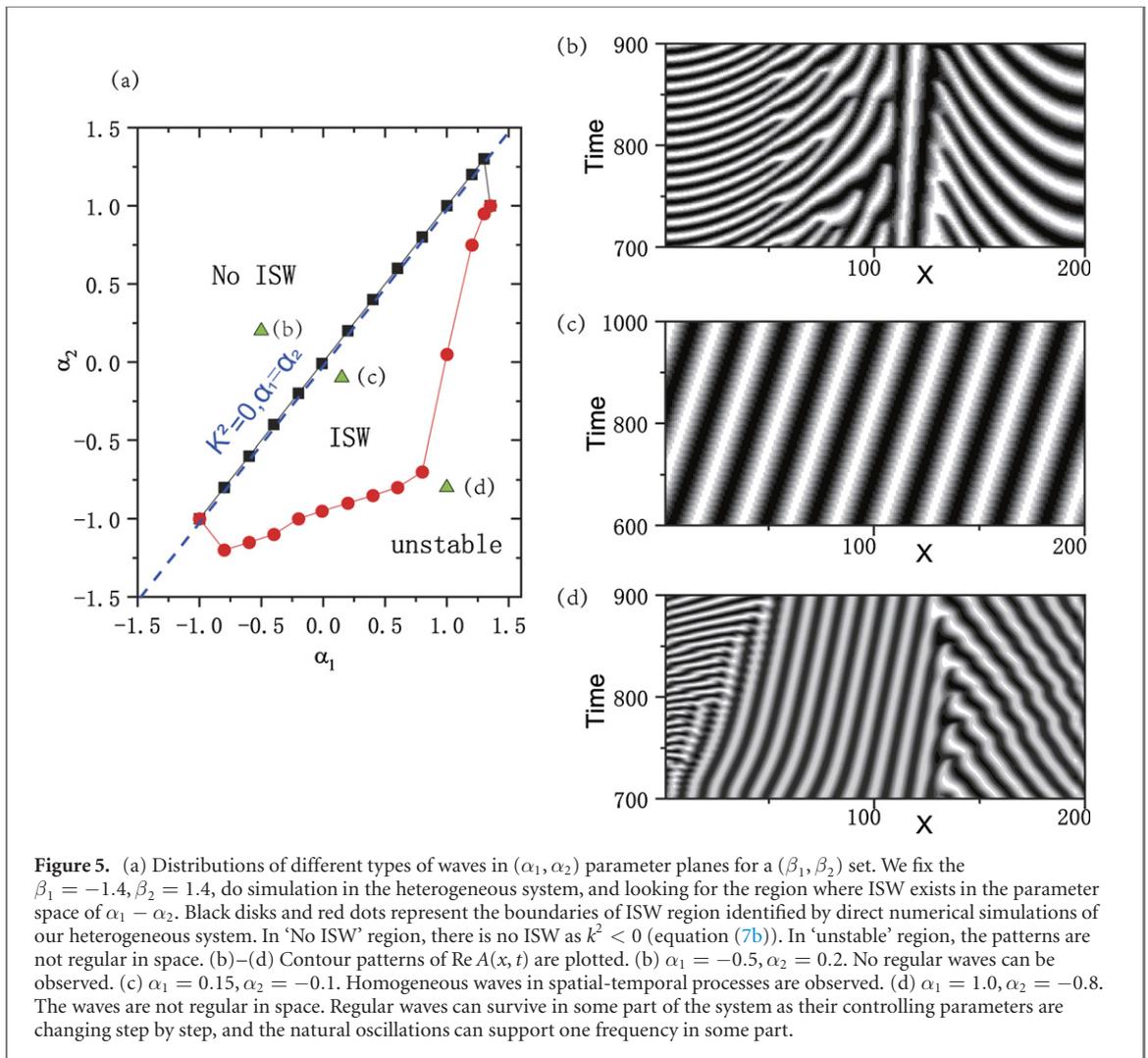
$$\omega(x) = \alpha(x) + (\beta(x) - \alpha(x)) * k(x)^2 \quad (7a)$$

$$k(x)^2 = k_0^2 = (\alpha_1 - \alpha_2) / (\beta_2 - \beta_1 - \alpha_2 + \alpha_1) \quad (7b)$$

$$\omega(x) = \omega_0 = (\alpha_1\beta_2 - \alpha_2\beta_1) / (\beta_2 - \beta_1 - \alpha_2 + \alpha_1) \quad (7c)$$

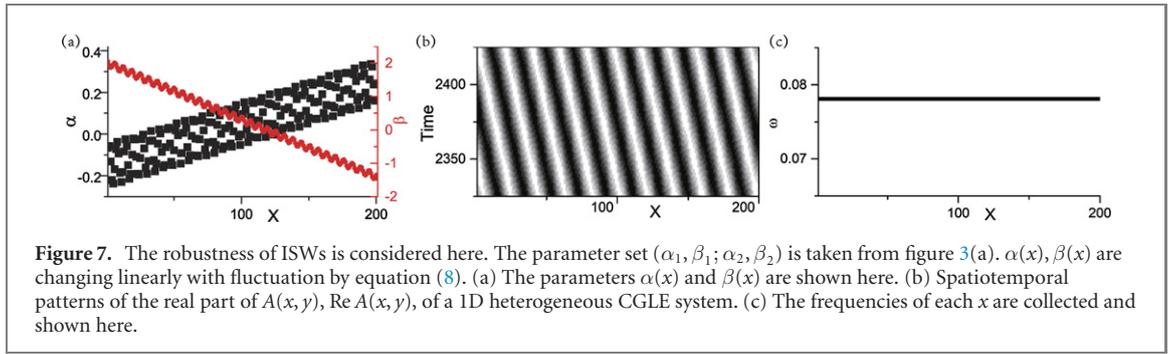
where (α_1, β_1) and (α_2, β_2) are the parameter sets at the beginning and ending of the heterogeneous system, respectively. In figure 4(a), we specify the $\omega(x) - k(x)$ relations in $\omega - k^2$ planes [the parameters are taken from figure 3(a)]. Each curve denotes a dispersion-relation of a parameter set (α, β) . The dispersion-relation curves are plotted for $(\alpha(x), \beta(x))$ with $x = 1, 20, 40, 60, \dots, 200$. The figure clearly shows that the curves overlap at the same point, noted as (ω_0, k_0^2) . We compare the numerical results with this point (ω_0, k_0^2) and find the two results coincide with each other. In other words, the inherent oscillations of each point compete and evolute. Finally, the wave train with wave frequency ω_0 and wavenumber k_0 survives in the heterogeneous system. It is similar to the results we found in the two-medium CGLE system. When the dispersion-relation curves of the two homogeneous media, which compose a two-medium system, intersect at one point (ω_0, k_0^2) , and this (ω_0, k_0^2) is a normal wave train in one medium with an anti-wave train on the other side, an ISW train can form from the interface and spread to the whole two-medium system [7]. For simplicity, this kind of homogeneous wave observed here is also called ISW, as (ω_0, k_0^2) is a normal wave train in one side while an anti-wave train on the other side.

In figure 4(b), we plot the dispersion-relation curves of parameters taken from figure 3(b). The dispersion-relation curves intersect at $\omega_0' = 0.414$, $k_0'^2 = 0.143$, which indicates the heterogeneous media can



support a homogenous wave train with that (ω_0, k_0) . However, the result we show in figure 3(b) is not consistent with this, and the simulation shows the frequency of waves is 0.243, which is close to α_2 . If we set an external wave source with this (ω'_0, k'_0) , a corresponding homogenous wave train will absolutely form in the heterogeneous media. Without an external wave source, this wave cannot be generated as there is no inherent source either. The final wave train is generated by the inherent oscillation $\alpha_2 = 0.25$ as the $(\alpha(x), \beta(x))$ sets here are all normal waves, and α_2 is the largest [23, 24].

To illustrate the conditions under which this kind of homogenous wave train (ISWs) can exist, we show an example in figure 5(a). We fix the $\beta_1 = -1.4, \beta_2 = 1.4$, do simulation in the heterogeneous system, and



looking for the region where ISW exists in the parameter space of $\alpha_1 - \alpha_2$. In figure 5(a), ISW in the region enclosed by disks, which is called ISW region. The lines with $k_0^2 = 0$ and $\alpha_1 = \alpha_2$ are the theoretical boundaries of ISW predicted by equation (7). In the ‘No ISW’ region, there are no ISWs due to violations of conditions of $k_0^2 < 0$. Contour pattern of this condition is shown in figure 5(b). There are no regular waves in spatio-temporal processes. The contour pattern of ISW region is shown in figure 5(c), regular waves are observed. In the ‘Unstable’ region, ISWs cannot survive due to the instability (there is a limit of frequency in each homogeneous media. Beyond the limitation, there are no stable waves). Contour pattern of $\text{Re } A(x, t)$ is shown in figure 5(d) on this condition. The ‘ISW’ region is similar to that of the two-medium system but is slightly larger. This result indicates that the global heterogeneity will broaden the region of ISWs and will benefit the generation of ISWs.

5. The robustness of ISWs

The spatial parameter of media changing linearly is a strict requirement that may not be easy to obtain. Here, we make the parameter sets go nonlinearly and check the existence of this kind of ISW. $\alpha(x), \beta(x)$ are set in equation (4) by $n = 2$, or 3. The asymptotic patterns are shown in figures 6(a) and (b). The spatial heterogeneity still seems transparent in pattern formation, and both wave frequencies are 0.0763, which is the same as the parameters with $n = 1$ in figure 3(a).

We add fluctuation to the system to check the robustness of the ISWs. The parameters of the system are set as follows.

$$\alpha(x)' = \alpha(x) + f * \sin(x) \quad (8a)$$

$$\beta(x)' = \beta(x) + f * \cos(x). \quad (8b)$$

The parameters α and β change via position x as shown in figure 7(a), and the lines of the parameter are no longer smooth. Here, the strength of the fluctuations is set as $f = 0.1$. The pattern of the systems is shown in figure 7(b). The frequencies of each point are all 0.0781, slightly larger than that without fluctuation, shown in figure 7(c). This result clearly shows the same properties in time. The pattern can maintain regularity in spatial space if the strength f decreases.

Together with the example of figures 6 and 7, we can deduce that the generation of a wave train with the same frequencies by spatial heterogeneity is not difficult, as the parameter sets are not limited to change linearly. The wave frequencies and wavenumbers can be theoretically predicted when parameters change regularly (as $n = 1, 2$, or 3), but not for conditions with fluctuation.

6. Conclusion

In conclusion, heterogeneity is a common property that can emerge in real experimental oscillatory media. Thus, the competitions between different waves and pattern formations in heterogenous systems are important and significant in practical applications. The selection of wave patterns in a global 1D heterogeneous CGLE system is studied and discussed in this paper.

We have studied system evolution under different parameter sets. The parameter sets have been set as changing linearly regularly, nonlinearly regularly, and linearly with fluctuations. We have found that it is easy to obtain a wave train with the same frequencies. First, when the parameters of each point can intersect at one point on the $\omega - k^2$ plane, and the frequencies of this point are NW on one side but AW in the other side, ISW can form. Second, the frequencies and wavenumbers of ISW can be explained perfectly. Third, the region of ISW under global heterogeneity has been shown in figure 5(a) and are slightly larger than that in

the two-submedium system. Finally, the generation of ISWs is robust under fluctuation as shown in figure 7.

The main conclusion as follows: (i) spatial heterogeneity in oscillatory media with random initial conditions can induce homogeneous planar wave trains. (ii) This kind of homogeneous wave train has similar properties to the interface-selected waves found in a two-medium system. The frequency and wavenumber can be precisely predicted under certain conditions. (iii) This kind of wave train can be found in a wide region in parameter space. (iv) This kind of wave train is robust with parameters varied nonlinearly or varied linearly with fluctuation.

The investigations in this paper are found in 1D systems with global spatially varied controlling parameters; however, regular patterns can also be observed generally in high-dimensional systems. Wave competitions and pattern formations in 2D oscillatory systems are of much more importance, as the types of waves become much richer, including spirals and anti-spirals, which can be self-sustained. We emphasize here that the results obtained are robust for heterogeneous oscillatory media without external pacing. The studies presented here are just the tip of the iceberg, and many interesting characteristic features and available applications of heterogeneous systems still require extensive exploration and explanation.

One of the expected developments of the present topic is experimental realization. The discussion in CGLE is not convenient for experimentalists directly. On the contrary, the analysis on reaction–diffusion systems may suggest the possible conditions of actual parameters, and give some easy guidance to experiments. The CGLE is deduced from reaction–diffusion system, and can be revised to real reaction–diffusion equation, like Brusselator equation [25]. And the negative refraction which is observed by simulation in a two-medium CGLE system [9], have been observed in the chlorite-iodidemalonic acid (CIMA) reaction system [8]. We discuss the global heterogeneous setting here can broaden the generating conditions of interface-selected waves. The global heterogeneous setting in real systems can be different light intensity in BZ reaction or different PVA concentrations in CIMA [8], and so on. We believe the phenomena found here can provide new insight in understanding the characteristics of nonlinear waves. Further investigation in this direction may greatly broaden our knowledge of pattern formations, competitions, and control of oscillatory waves.

Acknowledgments

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